Homework 4

---

1. Oscillatory Behavior: The general solution to the spring-block oscillator we found in class to be:

$$r_x(t) = A\sin\omega t + B\cos\omega t \qquad (*)$$

where $\omega = \sqrt{\frac{K}{m}}$, K is the spring constant and m is the mass of the block; and A and B are constants to be determined through initial conditions. Let the initial conditions be given in general through: $r_x(t=0) = r_{x0}; \quad v_x(t=0) = v_{x0}$.

(i) Study (and adapt if necessary) the MATLAB m-file spring_block_comp.m supplied and run it for the following case: spring constant K=15N/m, mass m=2kg,and initial conditions $r_x(t=0) = 0m; \quad v_x(t=0) = -0.2\,m/s$. Provide plots for both the analytical and the numerical (using the forward Euler algorithm and the Runge-Kutta method) results for the position and the velocity of the block versus time for a total of four periods of oscillation. [5 points]

(ii) Define the normalized error as:

$$error(t) = \frac{numerical - analytical}{amplitude}$$ where $E = \sqrt{A^2 + B^2}$ is called the amplitude of oscillation.

Plot the error versus time for the case given in (i) using time steps which are a tenth, a hundredth and a thousandth of the period of oscillation. Comment on your results. [5 points]

2. Exponential Growth/Decay: There are many physical phenomena where the rate at which something changes at any given time is directly proportional to the total amount of the same thing at that particular instant. For instance, the rate at which your bank account at any moment increases may be proportional to the amount of money that you have at that moment (this is *continuous* compound interest). The rate of increase of population of an ant colony may be proportional to the number of ants in the colony. Another example is the rate of decrease (decay) of a radioactive material which is proportional to the amount of that material at any particular instant. There are several other phenomena which exhibit such behavior.

That is, all these phenomena are given by a differential equation of the form:

$$\text{DE: } \frac{d\psi}{dt} = \kappa\psi \, ; \qquad \kappa \text{ is a given constant}$$

where $\psi(t)$ stands for your bank balance, or the population of ants, or the amount of radioactive material etc.

Suppose that the initial condition is given as: IC: $\psi(0) = \psi_o$

(2.i) [3 points] Show that the analytical solution to this DE and IC is:

$$\psi(t) = \psi_o e^{\kappa t}$$

Remark: If κ were positive, then this says that the quantity $\psi(t)$ increases **exponentially** with time (such as for your bank balance hopefully, unless your bank gives you a negative interest rate!). If κ were negative, then the quantity $\psi(t)$ decreases exponentially with time (such as for the decay of a radioactive material). The parameter κ has units of s^{-1}.
Its inverse t* = |1/κ| is called the **time constant** of the growth process (or decay process, if κ is negative). For exponentially increasing phenomena, this is the time when the quantity ψ increases to e times its original value. (Recall: e=2.71828). For exponentially decreasing phenomena, the time constant is the time required for the quantity to decrease to 1/e of its original value.

(2.ii) [3 points] Consider an ant colony started by a hardy bunch of pioneer ants numbering just 4 at time t=0 when the colony is founded. Finding themselves in an area of abundant food supply, and with no ant-eaters of any kind in sight, they begin an exponential growth process with a time constant of 1 week. In how many weeks will the colony grow to a population of 8,192?

(2.iii) [10 points] Write (modify) a MATLAB m-file using the Forward Euler and the Runge-Kutta algorithms to calculate the growth rate of the ant colony numerically. Plot the ant population as a function of time for each of the numerical methods, and calculate a normalized measure of error = {numerical_value(t)-analytical_value(t)}/initial_value for each algorithm. Comment on the accuracy of your results.

(2.iv) [4 points] For the parameters given in part (ii), calculate how long it takes for the ant colony to double in size from 4 to 8, and then from 8 to 16. How much longer will it take for the colony to double in size from 8192 to 16,384?

Aside: Do you think that the ant colony can sustain such growth for ever?