V IMPULSE AND MOMENTUM

V.1 Linear Impulse:
Newton’s laws say that if a net force $F$ acts on a body, then:

$$F = \frac{dp}{dt}$$

(5.1)

where $p = mv$ is the linear momentum of the body, where $m$ is the mass and $v$ is the velocity of the body.

Integrating the above with respect to time:

$$\int_{t_1}^{t_2} F dt = \int_{t_1}^{t_2} dp = p_2 - p_1$$

(5.2)

The left side is called the linear impulse due to the force $F$ over the time interval $t_1$ to $t_2$:

$$I_F = \int_{t_1}^{t_2} F dt$$

(5.3)

Remarks:

(i) The units of linear impulse in SI are N.s.

(ii) Often, such as during impact of bodies, it is not possible to measure the force of impact, but it is possible to obtain an average measure of the force by measuring its momentum before and after the impact through:

$$F_{av} = \frac{1}{t_2 - t_1} I_F = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} F dt = \frac{1}{t_2 - t_1} \{p_2 - p_1\}$$

(5.4)

Figure 5.1: Some representative impulsive forces.

(iii) The statement (5.2) says that the linear impulse imparted to an object is equal to the resulting change in linear momentum of the object.

(iv) If there is no net force acting on a body, clearly its linear momentum is unchanged.

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As we did when we discussed Newton’s laws, we can extend these ideas to a system of \( N \) particles. Let the \( i \)th particle have a mass \( m_i \), and be located at position \( \mathbf{r}_i \), moving with velocity \( \mathbf{v}_i \) with respect to some chosen coordinate system. Let us consider the forces on the \( i \)th particle in two parts: \( \mathbf{f}_{ij} \) is the force on the \( i \)th particle exerted by the \( j \)th particle in the system, and \( \mathbf{F}_i \) is the force exerted on the \( i \)th particle by something external to the system.

\[
\sum_j \mathbf{f}_{ij} + \mathbf{F}_i = \frac{d\mathbf{p}_i}{dt} \tag{5.5}
\]

There are \( N \) such equations, one for each particle. Suppose we sum all these \( N \) equations together:

\[
\sum_i \sum_j \mathbf{f}_{ij} + \sum_i \mathbf{F}_i = \sum_i \frac{d\mathbf{p}_i}{dt} \tag{5.6}
\]

Clearly the first term on the left side is zero (from Newton’s third law \( \mathbf{f}_{ij} = - \mathbf{f}_{ji} \)). Integrating the above with respect to time, we have:

\[
\int_{t_1}^{t_2} \sum_i \mathbf{F}_i dt = \mathbf{P}_2 - \mathbf{P}_1 \tag{5.7}
\]

where the linear momentum of the total system is given by:

\[
\mathbf{P} = \sum_{i=1}^N \mathbf{p}_i = \sum_{i=1}^N m_i \mathbf{v}_i \tag{5.8}
\]

If there is no net external force acting on a system of particles, then the above says that the linear momentum of the system is conserved.

\[
\mathbf{P} = \sum_{i=1}^N m_i \mathbf{v}_i = \text{constant} \tag{5.9}
\]

This is called the law of conservation of linear momentum for a closed system with no external forces acting on it. Note that the above is a set of three equations (one for each direction). It is useful to observe that if there is no external force along some particular direction, then the linear momentum associated with that particular direction is conserved, even if there are forces acting perpendicular to this direction.

Just as the law of conservation of energy is a universal truth, so is conservation of linear momentum. It holds, with appropriate interpretation, even at the subatomic level where quantum mechanics (and not classical Newtonian mechanics) rules. There is also a law of conservation of angular momentum, but we will not go into it here.
**Example: Stupid dog that runs off a raft**

A 40kg dog starts running on a 100 kg raft shown above at a relative speed of 5m/s and runs off the right end of the raft. What is the motion of the raft?

First, let's clarify a few things:
- the dog and the raft are initially at rest;
- horizontal forces between the raft and the water are neglected (this means the dog and the raft act as a closed system with no significant external forces in the horizontal direction);
- the dog runs at a constant speed of 5m/s and this is the relative speed of the dog with respect to the raft -- this is important since the raft will be moving as a result of the dog’s motion.

We will denote this velocity as:

\[
v_{D/R} = v_D - v_R\\ (*)
\]

where \(v_D\) and \(v_R\) are the velocities of the dog and the raft with respect to a fixed coordinate system (say the shore) and \(v_{DR} = 5\text{m/s}\).

Since the dog and raft are initially at rest, the initial velocities of both are zero and applying conservation of linear momentum, we obtain:

\[
0 = m_Dv_D + m_Rv_R\\ (**)
\]

Solving (*) and (**) for the velocity of the raft, we find:

\[
v_R = -\frac{m_D}{m_D + m_R}v_{D/R}
\]

which works out to \(-1.43\text{m/s}\) for the given values. The minus sign says that the boat moves in the direction opposite that of the dog’s motion. Note that the speed of the dog relative to the shore is only \(5\text{-}1.43=3.57\text{m/s}\). Thus the dog running to the right on the raft propels the raft to the left at a speed of \(1.43\text{m/s}\). The dog as it runs off the raft into the water is going at a speed of \(3.57\text{m/s}\).

Note that there is no time involved here - as soon as the dog begins running, these velocities are instantaneously obtained (at least in the ideal world with no friction, waves, etc). Actually no dog can instantaneously go from zero speed to some finite speed; dogs like everything else, have mass (even thin starving ones!) and they will need to accelerate.
V.2 Collisions:
The principles of conservation of linear momentum and conservation of mechanical energy provide us with a simple way to analyze bodies that collide. Let us consider the collision of two bodies A and B. Their masses are $m_A$ and $m_B$ and they move initially with velocities $v_A$ and $v_B$ respectively.

V.2.1 Perfectly plastic collisions: Here the masses stick together after the collision. From conservation of linear momentum, we have:

$$m_Av_A + m_Bv_B = (m_A + m_B)v_{AB} \quad (5.10)$$

where $v_{AB}$ is the velocity of the center of mass of the combined body after collision. Note that $v_{AB}$ can be calculated from the above.

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**Example 1:** Consider two balls of equal mass $m$ one of which is at rest, and the other moves initially along the x-direction with speed $v_o$ when it collides with the other. The two balls stick together upon impact. There are no external forces acting on the balls. What is the velocity of the combined ball after impact?

$$m v_o i = 2 m v_{AB} \Rightarrow v_{AB} = \frac{1}{2} v_o$$

Note that the total kinetic energy before collision: $K_{\text{before}} = \frac{1}{2} m v_o^2$

And the kinetic energy after collision is: $K_{\text{after}} = \frac{1}{2} (2m) \left[ \frac{v_o}{2} \right]^2 = \frac{1}{4} m v_o^2$

Note that for perfectly plastic collisions, the total kinetic energy is not conserved. So, energy is lost as heat in a perfectly plastic collision.

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V.2.2 Perfectly elastic collisions: Bodies are said to collide perfectly elastically if the total mechanical energy of the system is conserved as well. Since there are no external forces acting on the system (and assuming that there are no devices such as springs in which potential energy can be stored) this means that the kinetic energy of the system is conserved.

Let us denote the velocities after the collisions by a prime (‘). Then, conservation of linear momentum says:

$$m_Av_A + m_Bv_B = m_Av'_A + m_Bv'_B \quad (5.11)$$

and conservation of kinetic energy says:
\[
\frac{1}{2} m_A |v_A|^2 + \frac{1}{2} m_B |v_B|^2 = \frac{1}{2} m_A |v_A'|^2 + \frac{1}{2} m_B |v_B'|^2
\] (5.12)

which together provide the necessary equations to solve for the velocities after collision. For simplicity, we will confine attention to **direct collisions** where the bodies all travel along a straight line before and after collision.

**Example:** Consider two balls of equal mass \(m\) one of which (ball B) is at rest, and the other (ball A) moves initially along the x-direction with speed \(v_o\) when it collides with the other. The two balls collide perfectly elastically. There are no external forces acting on the balls. What are the velocities of the balls after impact?

Since the balls initially had no y- or z-velocity components, and since there are no external forces acting, clearly they will have zero y- and z-velocity components after impact as well.

So applying the conservation of linear momentum in the x-direction only provides:

\[mv_o = mv_{A'} + mv_{B'}\]

Conservation of energy gives:

\[\frac{1}{2} mv_o^2 = \frac{1}{2} m |v_A'|^2 + \frac{1}{2} m |v_B'|^2\]

Solving, we find that: ball A comes to a halt and ball B moves with speed \(v_o\).

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**V.2.3 Real collisions:** Most often collisions are neither perfectly plastic nor perfectly elastic. The very fact that we can hear two bodies collide means that some of their kinetic energies before collision has been transferred to sound energy. Usually, the bodies may undergo plastic deformation which leads to energy being lost as heat as well.

Consider the perfectly elastic direct collision of two bodies. We can now write conservation of linear momentum and energies as:

\[m_A v_A + m_B v_B = m_A v_A' + m_B v_B'\]
\[\frac{1}{2} m_A v_A^2 + \frac{1}{2} m_B v_B^2 = \frac{1}{2} m_A v_A'^2 + \frac{1}{2} m_B v_B'^2\]

The above can be re-written as:

\[m_A \{v_A - v_A'\} = m_B \{v_B' - v_B\}\]
\[m_A \{v_A^2 - v_A'^2\} = m_B \{v_B^2 - v_B'^2\}\]

Dividing one by the other and rearranging, we have:

\[\frac{v_B' - v_A'}{v_A' - v_B} = 1\]
For a perfectly plastic collision, we have $v_B' = v_A'$, and so:

$$\frac{v_B' - v_A'}{v_A' - v_B'} = 0.$$ 

This motivates us to define a **coefficient of restitution** $e$ as:

$$e = \frac{v_B' - v_A'}{v_A' - v_B'}. \quad (5.13)$$

The coefficient of restitution is 0 for a perfectly plastic collision, and 1 for a perfectly elastic collision. For collisions which are neither, we expect that the coefficient of restitution will take intermediate values, in which case (5.13) provides a relation between the before and after speeds in lieu of the one provided by conservation of energy (which will not hold if collisions are not perfectly elastic.)