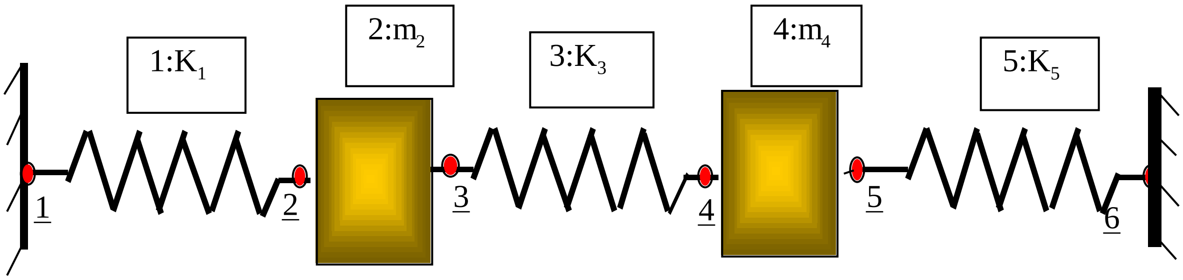
VI.7 Weakly-Coupled Oscillators

Consider the spring-mass-spring-mass-spring system shown below:



We want to investigate the behavior of this system as the spring constant K_3 of the coupling spring element 3 is varied. We will call the spring1-mass2 combination the *left oscillator*, and the spring5-mass4 combination the *right oscillator*.
If there is no coupling spring, we just have two distinct spring-mass oscillators which will each have a natural frequency which we know how to determine. We are now going to watch the behavior of the two oscillators as they are coupled by the coupling spring. Of course, this is just a 3spring-2mass system, and we can analyze it as such. Let us do so first.

- The state variables for the first-order analysis are:

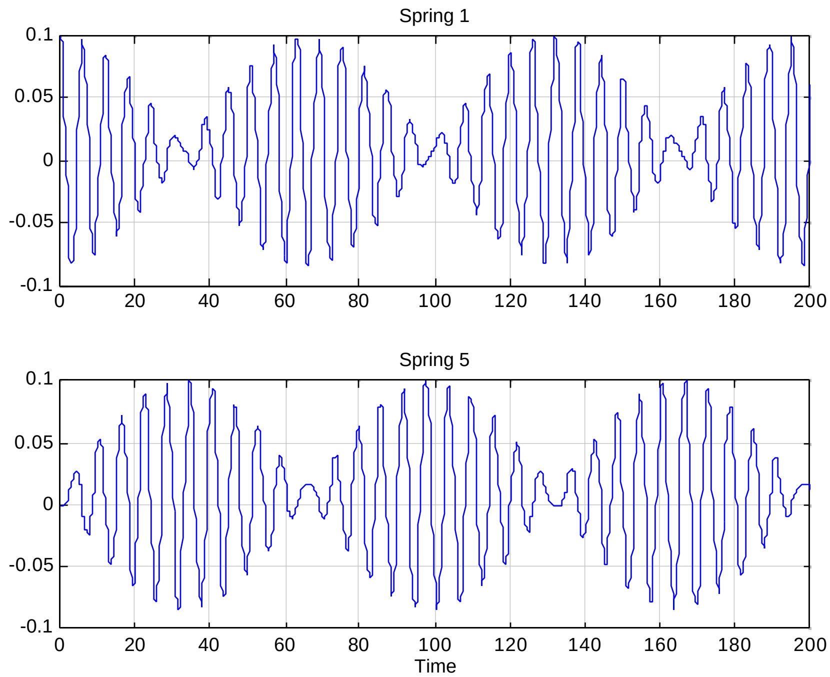
$$\underline{X} = \left\{ r_{sp1} \quad v_{m2} \quad r_{sp3} \quad v_{m4} \quad r_{sp5} \right\}^T$$

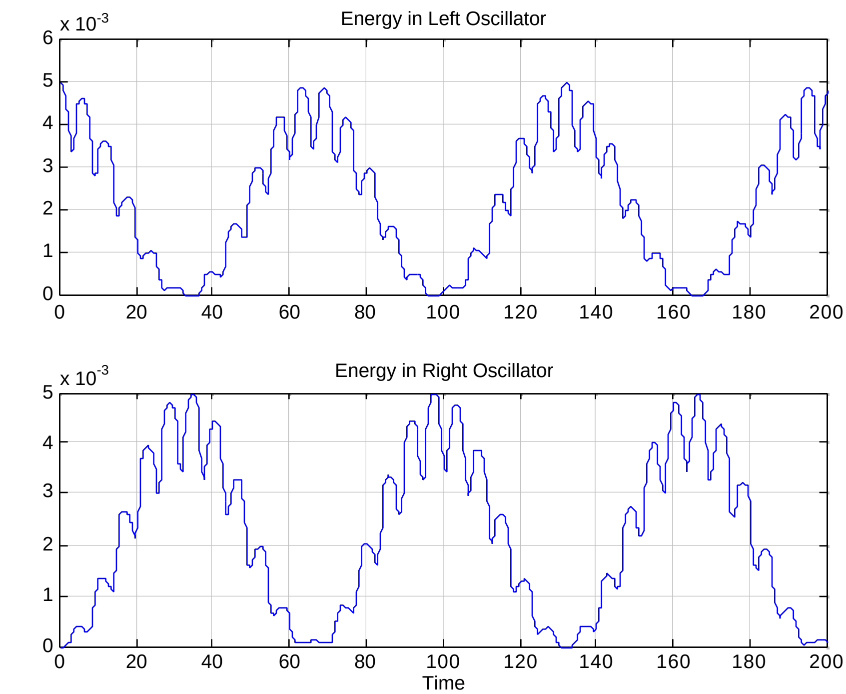
- Show that the first-order system equation is: $\dot{\underline{X}} = \underline{A}\,\underline{X}$

$$\text{where } \underline{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -\frac{K_1}{m_2} & 0 & \frac{K_3}{m_2} & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -\frac{K_3}{m_4} & 0 & \frac{K_5}{m_4} \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

- MATLAB code: *sd_weak_couple.m* along with *rate_fn.m*
- For each of the cases below:
  - Plot the stretches of the three spring elements
  - Plot the (potential + kinetic) energies in the left and the right oscillators and the potential energy in the coupling spring
  - Comment on the observed behavior

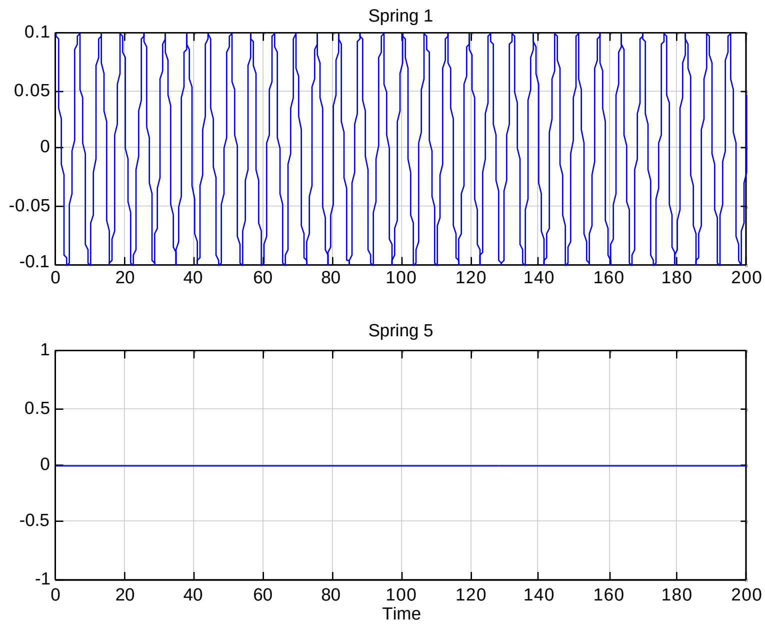
Case (a): Parameters: $K_1 = K_5 = 1 N/m$; $m_2 = m_4 = 1 kg$.
Initial conditions: $\underline{X}(0) = \{0.1 \quad 0 \quad 0 \quad 0 \quad 0\}^T$
Coupling spring constant $K_3 = 0.1$

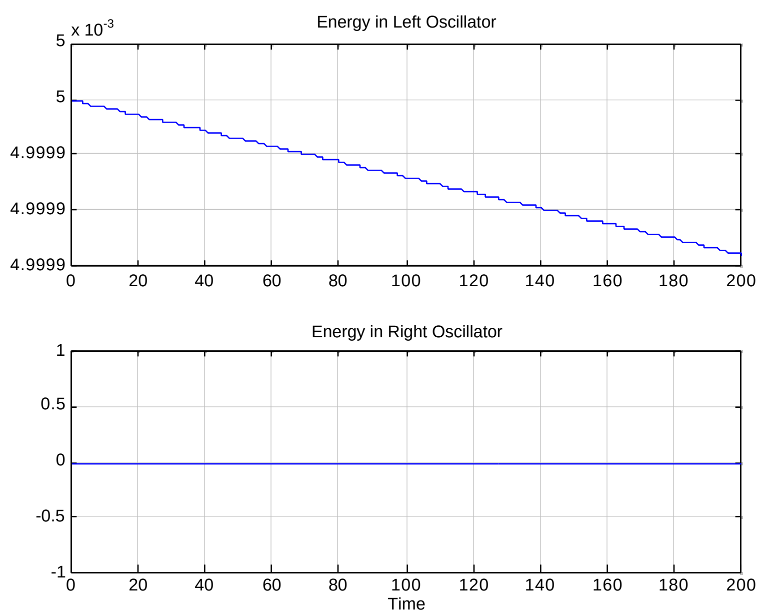




Comment: "Sloshing" behavior – Energy bounces from left to right oscillators back and forth facilitated by the weak coupling spring.

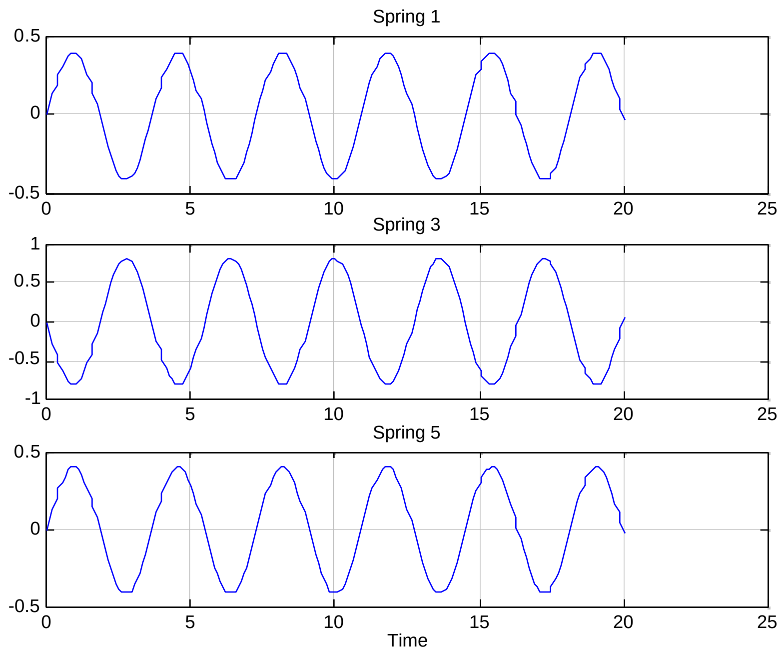
Case (b): Parameters: $K_1=K_5=1N/m$; $m_2=m_4=1kg$.
Initial conditions: $\underline{X}(0) = \{0.1 \quad 0 \quad 0 \quad 0 \quad 0\}^T$
Coupling spring constant K3 = 0

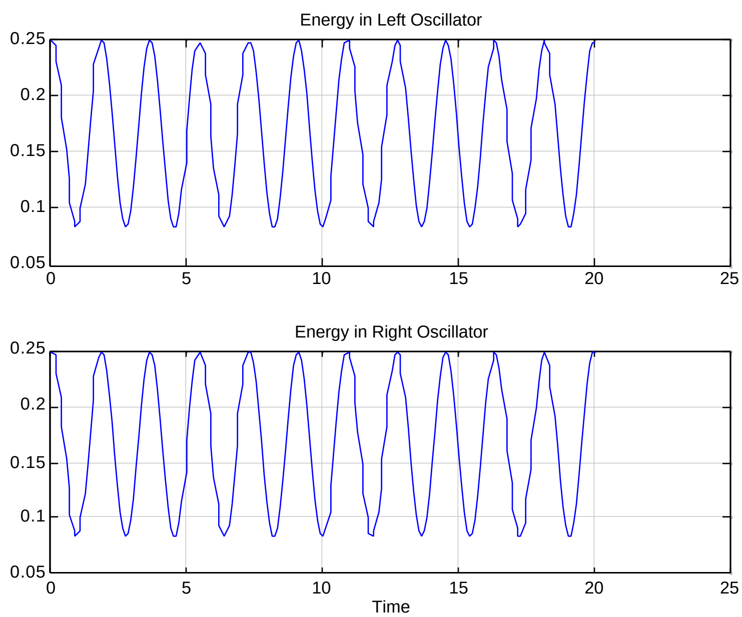




Comment: No coupling - Only left oscillator moves since we only set that off and there is no coupling. Apparent energy loss (look at the vertical axis scale) is due to very small numerical error. Can improve accuracy by making time-step smaller.

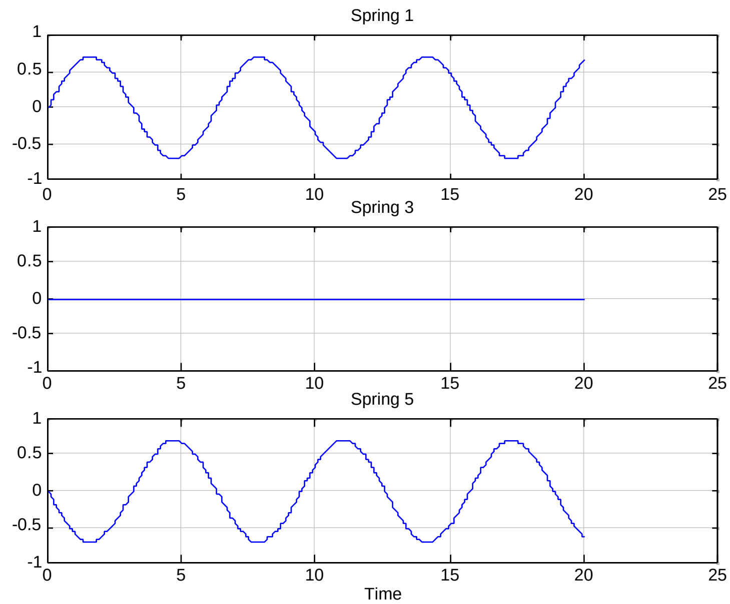
Case (c): Parameters: $K_1 = K_3 = K_5 = 1$N/m; $m_2 = m_4 = 1$kg.
Initial conditions: $\underline{X}(0) = \{0 \quad 0.7071 \quad 0 \quad 0.7071 \quad 0\}^T$

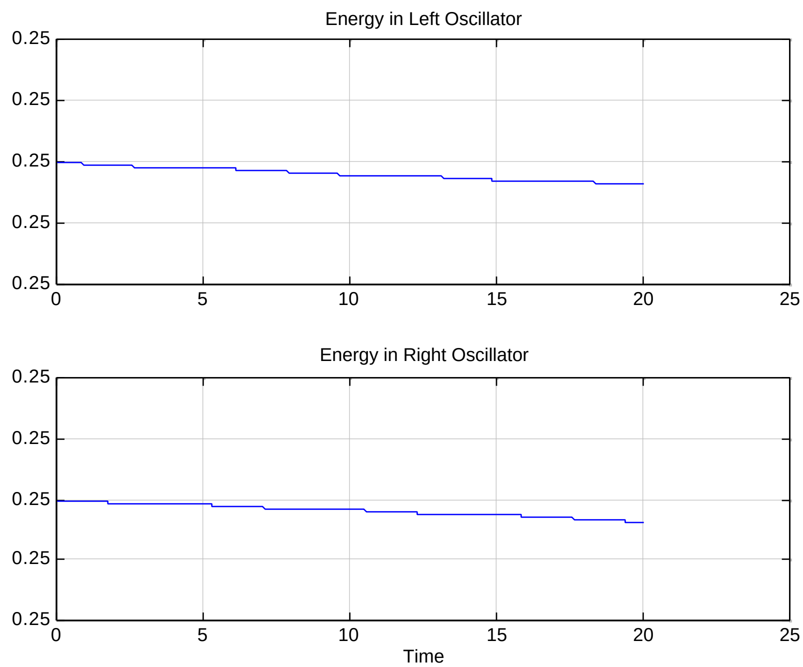




Comment: Normal mode - Looks like we have hit a normal with left and right oscillators in phase.

Case (d): Parameters: $K_1 = K_3 = K_5 = 1 N/m$; $m_2 = m_4 = 1 kg$.
Initial conditions: $\underline{X}(0) = \{0 \quad 0.7071 \quad 0 \quad -0.7071 \quad 0\}^T$





Comment: Another normal mode – interesting fact is this time the coupling spring does not stretch and so does not store any energy! The two oscillators here are out-of-phase. Also, note that the slight loss in energy is numerical error (I used dt=0.01 here).

Cases (c) and (d) appear to be normal modes of this system. Note that I have set the system off into these normal modes by imparting initial *velocities* to the masses

Let us obtain the *second-order system equations* in order to figure out its normal modes analytically.

We will choose the velocities of the masses as our reduced set of state variables.

$$\underline{Y} = \begin{bmatrix} v_{m2} & v_{m4} \end{bmatrix}^T$$

Second-order system equations: $\ddot{\underline{Y}} = \underline{\underline{B}}\,\underline{Y}$

where
$$\underline{\underline{B}} = \begin{bmatrix} -\dfrac{K_1}{m_2} - \dfrac{K_3}{m_2} & \dfrac{K_3}{m_2} \\ \dfrac{K_3}{m_4} & -\dfrac{K_3}{m_4} - \dfrac{K_5}{m_4} \end{bmatrix}$$

For each of the following cases,

- Determine the natural frequencies (eigenvalues) and the eigenvectors;
- Plot the corresponding normal mode oscillations (spring stretches and energies in the left and right oscillators);
- Comment on the observed behavior.

Case (e): Parameters: $K_1 = K_5 = 1$N/m; $m_2 = m_4 = 1$kg.
Coupling spring constant $K_3 = 0.1$

Ans: $\omega_1 = 1.0954$ and $\omega_2 = 1.0000$
The Eigenvectors are:

0.7071	0.7071
-0.7071	0.7071

Comment: If we start off the system with these initial velocities, the system goes into the corresponding normal mode. Since coupling spring constant is very small, the natural frequencies of this system are almost the same as those of the completely decoupled left and right oscillators.

Case (f): Parameters: $K_1 = K_5 = 1$N/m; $m_2 = m_4 = 1$kg.
Coupling spring constant $K_3 = 1$

Ans: $\omega_1 = 1.7321$ and $\omega_2 = 1.0000$
The Eigenvectors are:

0.7071	0.7071
-0.7071	0.7071

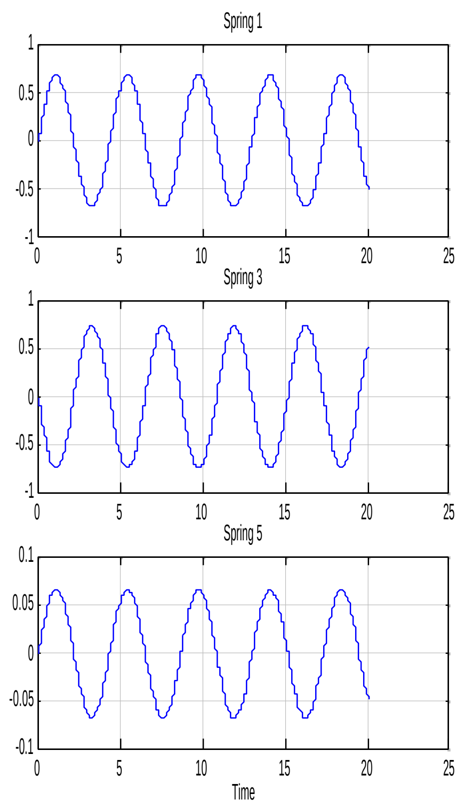
Comment: If we start off the system with these initial velocities, the system goes into the corresponding normal mode. The eigenvectors do not seem to depend on the coupling spring constant! Would be interesting to numerically experiment with other values of the coupling spring constant.

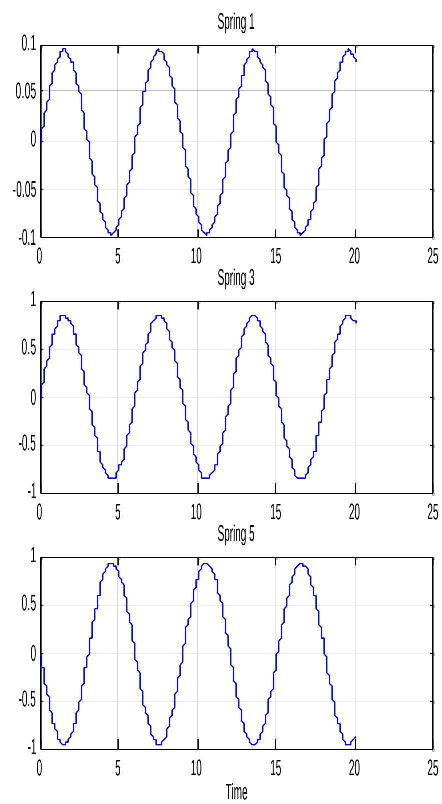
Let us now see what happens if we the natural frequencies of the decoupled oscillators are not the same.

Case (g): Parameters: K_1=2N/m K_5=1N/m; m_2=m_4=1kg.
What are the natural frequencies of the left and right de-coupled oscillators in this case? Ans: √2 radians/s and 1 radian/sec

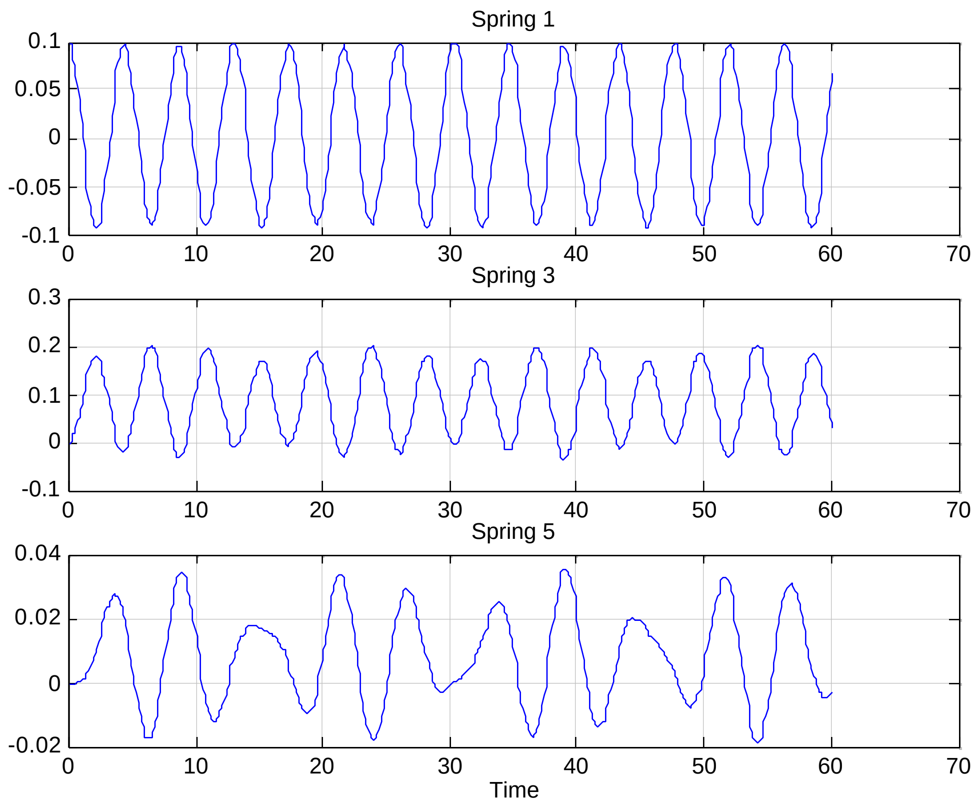
Case (h): Parameters: K_1=2N/m K_5=1N/m; m_2=m_4=1kg.
Coupling spring constant K_3= 0.1
What are the normal mode frequencies and eigenvectors for this case, and plot the asscociated motions.
The natural frequencies are: 1.4526 and 1.0441 and the associated igenvectors are:

0.9951	0.0985
-0.0985	0.9951





Case (i): Parameters: K_1=2N/m K_5=1N/m; m_2=m_4=1kg.
Initial conditions: $\underline{X}(0) = \{0.1 \quad 0 \quad 0 \quad 0 \quad 0\}^T$
Coupling spring constant K_3= 0.1



Comment: Do not observe sloshing behavior.

Case (j): Parameters: K_1=2N/m K_5=1N/m; m_2=m_4=1kg.
Initial conditions: $\underline{X}(0) = \{0.1 \quad 0 \quad 0 \quad 0 \quad 0\}^T$
Try different values for the coupling spring constant K_3 and see if you can get this system to "slosh" as it did in case (a). After numerically experimenting a bit, suggest what kinds of systems might slosh and under what conditions.