VII. SYSTEMS DYNAMICS – Transitioning to Other Domains by Analogy

We have thus far investigated the dynamics of *translational mechanical systems*.

We have encountered **dynamic variables** such as:

- *force,*
  (which is an "effort" variable because it causes motion)
- *velocity,*
  (which is a "flow" variable because it characterizes the motion)
- *displacement,*
  (which is also a flow variable)
- and we have looked at concepts of work done, energy stored, and power dissipated.

We have identified **elements** such as:

- *springs,*
- *dampers, and*
- *masses,*

each with *one* defining property. We saw that these basic idealized elements can be combined to make up complex translational mechanical systems.

We have looked at various interesting dynamic **system behavior** such as:

- *exponential decay or growth,*
- *oscillatory motions,*
- *damped oscillations,*
- *normal mode response and resonance of systems, and*
- *sloshing motions of weakly-coupled oscillators.*

In the process we have acquired some new **tools of mathematics** as well:

- *numerical solution of differential equations,*
- *analytical solution of Euler's differential equations, and*
- *eigenvalue analysis.*

Table V11.1 summarizes the various elements of the translational mechanics domain, the parameters associated with each, their defining characteristic, and their energy storing or dissipating effect.

Now it is time for us to put the knowledge we have gained to good use *elsewhere* in other domains of physics dealing with: rotational mechanical systems, electrical systems, hydraulic/pneumatic systems, and thermal systems. In the rest of this course, you will begin to appreciate the similarities of system behavior across various physical domains. Nature, it seems, has a few bags of tricks that it likes to play over and over again in different venues. Now that we have seen some of these tricks in the translational mechanical domain, let us move on to other domains. One word of caution: we are going to rely quite heavily on *analogy* in trying to port our knowledge of one domain to others. In this process, it is possible for you to lose sight of the physical concepts of the various domains, especially since I will not dwell at any great length on these. In your subsequent physics and major courses, you will encounter these concepts again, and it would be a good idea to fully familiarize yourselves with their physical meaning at that time.

Let me start with the rotational mechanics domain because the concepts there are quite closely related to the translational mechanics domain, and you may actually be familiar with some of them from EA2. How we port our understanding from one domain to another by *analogy* will become clear now.

VII.1 ROTATIONAL MECHANICAL SYSTEMS

This domain deals with systems that rotate about some axis and it models behavior of physical systems such as torsional pendulums, twisting motion of buildings and bridges, and machines with rotary shafts. We will once again restrict attention to one-dimensional rotational systems where all the elements of the system rotate about the same axis, which for simplicity we shall assume passes through the center of mass of all the elements.

The **dynamic variables** of this domain are:

- moment or *torque*, T,
  which is an effort variable analogous to force; (units: N.m)
- *angular deflection* or *twist*, θ,
  which is analogous to displacement; (units: radians)[†] and
- *angular velocity*, $\omega = \dot{\theta}$, which is analogous to velocity; (units: radians/sec)

No doubt you are familiar with some of these entities from EA2.

[†] Note angles are *dimensionless*, and as such the radian is not really a unit, but you might want to keep track of units of physical quantities involving angles by denoting radians wherever an angle is involved just so the physics becomes apparent.

Newtons' second law for the rotational systems under consideration takes the form:

(Net torque on a body is proportional to its angular acceleration)

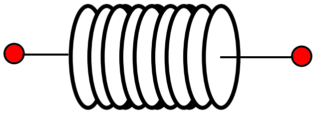
$$\sum T = I\dot{\omega}$$

where the proportionality constant I is called the *mass moment of inertia* of the body (units: kg.m^2), which is a property of the geometry and composition of the body[†].

The **elements** of the rotational mechanics domain are:

Torsional spring: The defining characteristic of this element is that it relates the torque acting on it to the relative twist of one end with respect to the other. Torsional springs are assumed to have no mass moment of inertia (remember, *one defining characteristic per element*). The torque on either end of a torsional spring must therefore be of equal magnitude and opposite sense. In drawing free-body diagrams, we will adopt the sign convention that a torque is considered positive if it acts clockwise looking into the element.

The symbol for a torsional spring will be:



Thus, the constitutive relation for a torsional spring element is:

$$T = k\theta$$

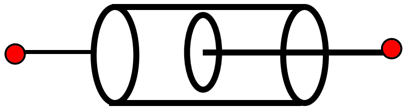
where T is the torque acting through the spring; θ is the *relative* twist of one end with respect to the other; and the *torsional spring constant* k has units of N.m/radian.

Examples of torsional springs: (a) look inside an old mechanical watch; (b) a piece of a wire that is subject to twisting), (c) an elastic shaft etc.

Rotational damper: The defining characteristic of a rotational damper is that it relates the torque acting through it to the relative *rate* of twist of its two ends. Rotary dampers are assumed to have no mass moment of inertia. Therefore the torque acting on either end of a rotary damper must be of equal magnitude and opposite sense.

[†] As we restrict attention to systems rotating about their centroidal axes, the torques are taken with respect to this axis, the angular velocities and accelerations are about this axis, and the mass moment of inertia is about this axis as well.

The symbol for a rotary damper is:



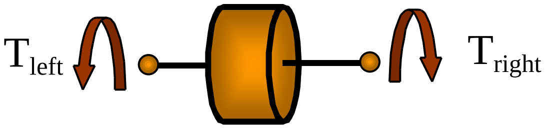
A rotary damper can be made from an oil-filled cylinder with a piston that rotates (hence the symbol)
The constitutive law for a rotary damper is:

$$T = c\dot{\theta} = c\omega$$

where *c* is the rotary damping coefficient (units: N.m.s)

<u>Rotary inertial element:</u> The last element of the rotational mechanical system is analogous to the mass (or block) of the translational mechanics domain. The only defining property of the rotary inertial element is its rotary inertia, also known as mass moment of inertia. Rotary inertial elements are rigid and so cannot twist under the action of torques.

The symbol for the rotary inertial element is:



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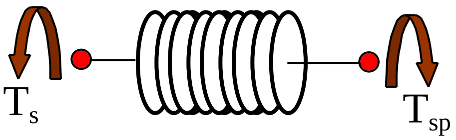
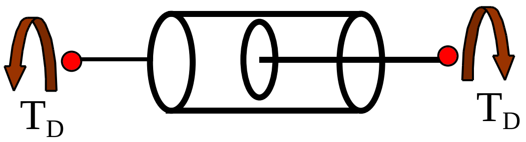
Unlike the other rotary elements, the torques on either side of a rotary inertial element need not be the same. However, since we have assumed that this element is rigid, the element can only rotate as a rigid body, ie. the angle of twist of the left and right nodes must be the same. From Newton's second law for rotary systems, we have:

$$T_{right} - T_{left} = I\dot{\omega}$$

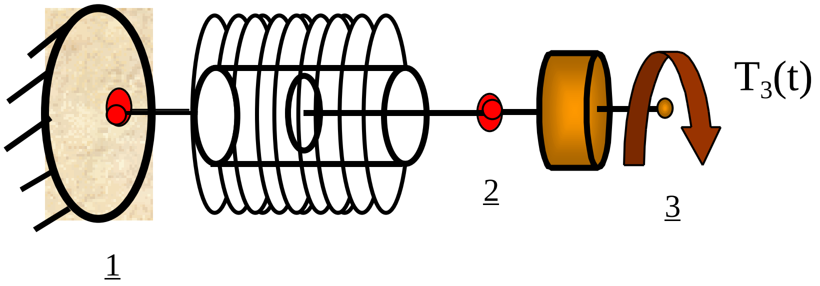
where *I* is the mass moment of inertia.

Sign convention: It is worth reiterating our sign convention for the torques and angles of twist. Positive direction will be taken as clockwise looking into an element from its nodes. As before, I will assume that the torques on all rotary spring and rotary damper elements are taken in the positive sense, and let the analysis dictate the actual sign. The sign convention for springs and dampers is therefore as given in these figures below. Note that the torques at the left and right nodes of these elements must be of equal magnitude but opposite sense (since they are assumed to have no mass moment of inertia).





Torsional-spring-damper-rotary inertia system:
Consider the rotary spring-damper-inertia system shown below. The rotary spring and damper are in parallel, ie they both must undergo the same twist. An externally applied torque source acts at node 3.



This could be a torsional pendulum (where the spring is a wire undergoing twist) and the damper models frictional losses in the material. It could also be a mechanical shaft being driven by some motor. Let us analyze this system.

State Equations: Pick the spring twist and the angular velocity of the rotary inertial element as the dynamic variables to track:

$$\underline{X} = \begin{Bmatrix} \theta_{sp} \\ \omega_I \end{Bmatrix}$$

Geometry:

$$(i)\ \theta_1 = 0$$

$$(ii)\ \theta_{sp} = \theta_2 - \theta_1 = \theta_2$$

$$(iii)\ \omega_D = \dot{\theta}_2 - \dot{\theta}_1 = \dot{\theta}_2$$

$$(iv)\ \omega_I = \dot{\theta}_3 = \dot{\theta}_2$$

It is clear that in this case the twists (and angular velocities) of all three components are equal. In a more general system, this might not be the case.

Newton's second law:

$$(v)\ T_3(t) - T_{sp} - T_D = I\dot{\omega}_I$$

Constitutive law:

$$(vi)\ T_{sp} = k\theta_{sp}$$

$$(vii)\ T_D = c\omega_D$$

First-order state equations:

$$\begin{Bmatrix} \dot{\theta}_{sp} \\ \dot{\omega}_I \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{I} & -\dfrac{c}{I} \end{bmatrix} \begin{Bmatrix} \theta_{sp} \\ \omega_I \end{Bmatrix} + \begin{Bmatrix} 0 \\ T_3(t)/I \end{Bmatrix}$$

Second-order state equations:

We choose $\theta = \theta_{sp}$ (equal to twist of the damper and the mass as well, in this case) as our reduced set of state variables. Then, you can reduce the first-order state equations to:

$$\frac{d^2\theta}{dt^2} + \frac{c}{I}\frac{d\theta}{dt} + \frac{k}{I}\theta = \frac{T_3(t)}{I}$$

is the second-order state equation. We have seen this equation before in connection with the damped spring-mass oscillator (but here with the addition of an external torque source). Clearly what we have here is a damped *torsional* oscillator with external torquing. We can immediately identify the behavior of this system for various cases by analogy with its counterpart in the translational mechanics domain.

System Behavior: Let us look at the system behavior for a few special cases. Note, in particular, the parallels between the translational spring-mass-damper and its rotational counterpart.

Case (a): No torque source, and no torsional damper:

$$\frac{d^2\theta}{dt^2} + \frac{k}{I}\theta = 0$$

By now you know that this kind of a system behaves in an oscillatory fashion:

$$\theta(t) = A\sin\Omega_n t + B\cos\Omega_n t,$$

where A and B are constants to be determined from initial conditions, and

$$\Omega_n = \sqrt{\frac{k}{I}}$$

is the natural angular frequency of oscillation of the system. (I have to use capital Ω_n so to not to confuse it with *angular velocity* ω of the various elements). What we have here is nothing more than an undamped torsional pendulum.

We can make an energy argument here analogous to the one we made for the translational spring-mass oscillator. The rotary inertia and the torsional spring exchange kinetic and potential energies back and forth, and since there are no energy dissipating elements in this case, the oscillations go on forever.

Case (b): No torque source, and no torsional spring:

$$\frac{d\theta}{dt} + \frac{k}{c}\theta = 0$$

the solution to which is:

$$\theta(t) = \theta_o e^{-\frac{k}{c}t}$$

which is an exponential decay starting from an initial twist of θ_o. Here the energy argument goes as follows: there is no inertial element here to store kinetic energy, and so the initial potential energy in the twisted spring is just slowly dissipated away in the damper as the spring unwinds to its relaxed state.

Case (c): No torque source, but with damper and spring:

$$\frac{d^2\theta}{dt^2} + \frac{c}{I}\frac{d\theta}{dt} + \frac{k}{I}\theta = 0$$

Analogous to the translational spring-mass-damper system, we could have either *overdamped* response which is just an exponential decay without any oscillations, or we could have a *damped oscillatory* response where the amplitude of oscillation becomes smaller and smaller as the damper steals away the potential and kinetic energies in the spring and rotary inertial elements.

Case (d): Harmonic torque source, with no damping:

$$\frac{d^2\theta}{dt^2} + \frac{k}{I}\theta = \frac{T_0}{I}\sin\Omega_f t$$

where Ω_f is the angular frequency of the applied torque source. Clearly, we can expect resonant behavior when the external driving frequency $\Omega_f \to \Omega_n$. Refer back to our earlier analysis of forced oscillations of a translational mechanical system.

And on and on and on. The parallels between the rotary mechanics domain and its translational counterpart are so complete that we can anticipate that we should be able to observe weak-coupling phenomena, resonance, normal mode behavior etcetera in complex systems of torsional springs, dampers and rotary inertial elements. We might as well say that we know quite a bit about the rotary mechanics domain and move on. (Not so fast! I will give you some homework assignments on rotary systems in due course.)

So, let us now move on to the electrical domain and see if we can:

(i) identify effort and flow variables (voltage; and current and charge);
(ii) figure out dynamic laws or rules that relate these variables (Kirchchoff’s loop and junction rules);
(iii) cook up idealized devices that store energy (capacitors), devices that dissipate energy (resistors), and devices that act like inertial elements (inductors);
(iv) make up more complex systems with these basic elements as building blocks; and
(v) investigate systems that exhibit phenomena such as: exponential decay or growth; damped or undamped oscillations; resonance and normal mode behavior; et cetera.