III The spring-block oscillator

Let us consider a simple spring-block system in the absence of any friction or air drag. We assume that the block is rigid and has mass, and the spring is massless but is deformable. This is what is called a lumped parameter idealization of a simple mechanical system, where all the "springiness" (deformability) is attributed to the spring and all the "massiveness" is attributed to the block. In reality, real springs do have mass, and real blocks are deformable. However, ours is a useful approximation (if, for instance, the mass of the block is a lot more than that of the spring, and the stiffness of the block is a lot higher than that of the spring). Also, we can actually model useful real life systems with such simple idealized elements.

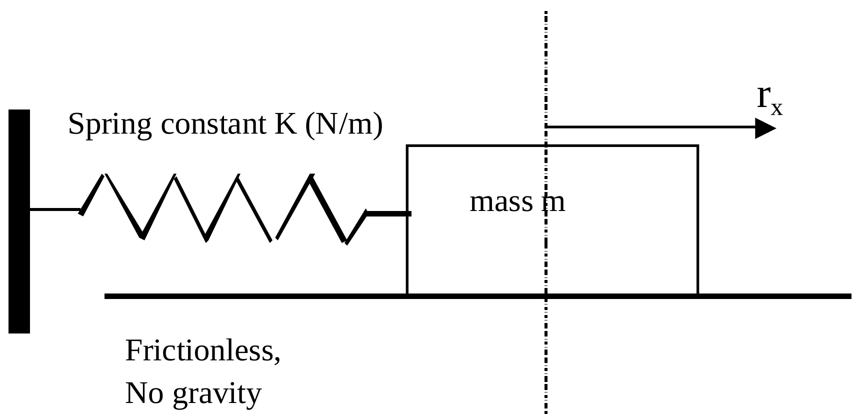

Figure 3.1: Undamped spring-block oscillator

Let the position of the block when the spring is unstretched ('relaxed' position of the spring) be the origin of our coordinate system (see Fig.3.1). If the block is now moved by an amount r_x as shown, the spring stretches (or compresses) and therefore starts to pull (or push) on the block with a force $\sum F_x = -Kr_x$ (we are considering a linear elastic spring which you have seen in EA2). From the free-body diagram of the block, we find that the equations of motion of block are given by:

$$\begin{aligned} \frac{dv_x}{dt} &= \frac{1}{m}\sum F_x = -\left[\frac{K}{m}\right] r_x \\ \frac{dr_x}{dt} &= v_x \end{aligned} \qquad (3.1)$$

Let us say that at time t=0, the block is gently moved to r_{x0} and let go with zero velocity at this time, ie $r_x(t=0) = r_{xo}$; $v_x(t=0) = v_{xo} = 0$. Given these initial conditions, we want to find the subsequent motion of the block.

You can modify one of the MATLAB m-files we have used previously to integrate the above numerically. MATLAB spits out the following graphs for the position and velocity of the block:

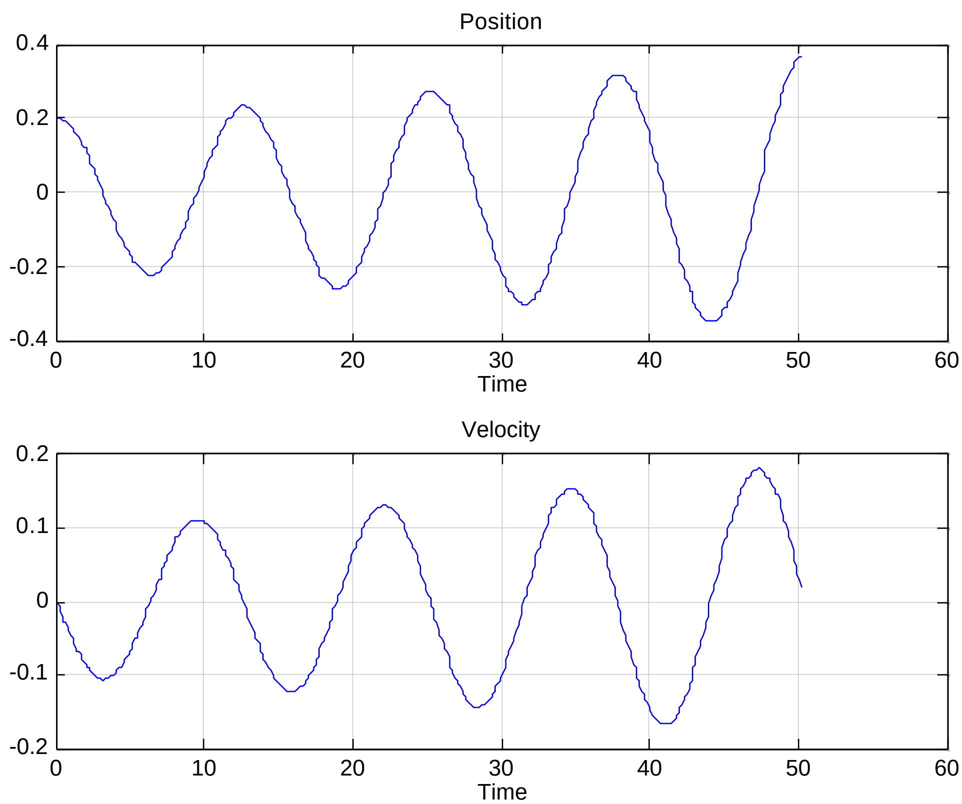

Figure 3.2: MATLAB solution for the position and velocity of the spring-block oscillator. Forward Euler algorithm used.

Remarks:

1. There is something fishy about this numerical solution. It appears that the oscillation amplitude increases with time! This has to be incorrect as it goes counter to what we see physically. If anything, the oscillations will die down in reality, but that has to do with friction and energy dissipation which we have not included here. As we will see

shortly, for our frictionless undamped spring-block system, theoretically the oscillations should go on and on forever, neither growing nor decreasing in amplitude.

2. The culprit for our erroneous solution turns out to be the numerical algorithm that we have been using so far. If you use a smaller time interval Δt, you will find that your solution gets better. The reason for this is not hard to see. We have been calculating the next position of the block by adding to its current position the *displacement* that the block will undergo over the interval Δt. But in doing this we have assumed that the velocity of the block remains constant during the *entire* interval Δt. Clearly, this need not be, and indeed even the solution for the velocity (fig.3.2) seems to indicate that this is not so. Indeed, no matter how small a time step we choose, we find that the numerical solution always indicates that the amplitude of oscillation will increase with time. Therefore it seems that we have to come up with a better numerical algorithm than the forward Euler algorithm we have used so far.

But before we come up with some better numerical integration schemes, let us see if we can analytically solve for the motion of this spring-block system.

III.1 Analytical solution for the undamped spring-block oscillator:

First let us rewrite the spring-block oscillator equations of motion (3.1) as a single equation:

$$\frac{d^2 r_x}{dt^2} = -\frac{K}{m} r_x \qquad (3.6)$$

This is a differential equation (DE) of motion where the unknown position of the block $r_x(t)$ appears on both sides. How do we solve for $r_x(t)$? Look at the DE (3.6) carefully. What does it say? It says that we want a function $r_x(t)$ which when differentiated twice yields the same function $r_x(t)$but multiplied by a constant (-K/m). Do you know of any function that might behave like this? Sure you do. Consider a simple sine function: $\sin\omega t$ where ω is some constant. When we differentiate a sine once, it becomes a cosine, and that is not good. But when you differentiate a cosine it turns back into a sine, and that is just perfect. Therefore differentiating a sine function twice provides a result that is also a sine function, possibly multiplied by some constants. Precisely what we want. This process of guessing a solution to a DE is the best way of solving it! Mathematically,

Seek: $r_x(t) = \sin\omega t$ as a trial solution to (3.6)

Then: DE(3.6)→ $-\omega^2 \sin\omega t = -\frac{K}{m}\sin\omega t$.

Which says that if we choose $\omega = \sqrt{\frac{K}{m}}$, then $r_x(t) = \sin\omega t$ satisfies the DE(3.6). Which is the same as saying that we have integrated the original equations of motion (3.1). But before you start celebrating, how do we know that this is the only possible solution? In fact, it is a simple matter to show that: $r_x(t) = A\sin\omega t$ where A is any arbitrary constant also satisfies the DE(3.6). Furthermore, if you are really into this, you would have figured out that $r_x(t) = B.\cos\omega t$ (B constant) would also satisfy the DE. And so would: $r_x(t) = A.\sin\omega t + B.\cos\omega t$ where A and B are constants. (Try it). In fact,

$$r_x(t) = A.\sin\omega t + B.\cos\omega t \qquad (3.7)$$

is the complete solution $r_x(t)$ for the DE(3.6). (You will learn more about this in EA4).

Now, what are these A and B constants? And how do we go about finding them? If you think about this a bit, you will see that we need to know two things (there are two constants after all) about the block at some time during its motion. Typically, this will be at the start of the motion of the block, ie we will need to use the *initial* conditions of the block.

For instance, if initially (at time t=0) the block has been gently moved to $r_x(t=0)=r_{xo}$, and then it is let go without imparting any velocity to it, ie. $v_x(t=0)=0$. Then, we find that: A=0 and B= r_{xo}, and so $r_x(t) = r_{xo}\cos\omega t$ gives the position of the block for all

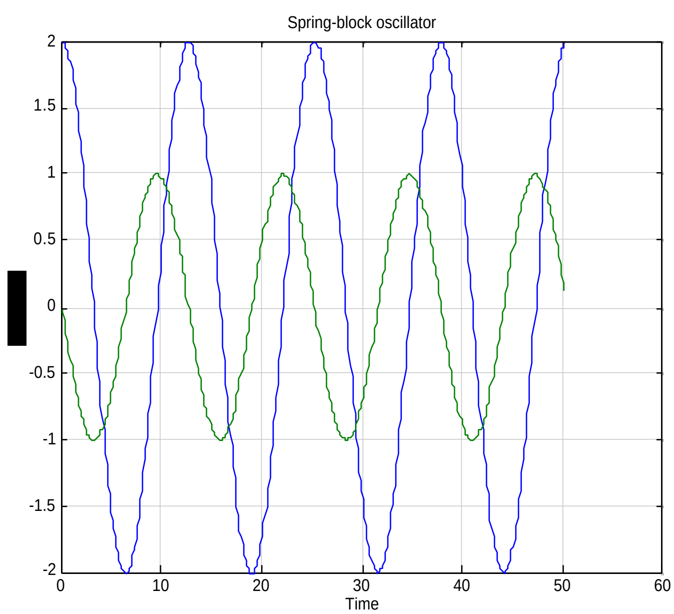

Figure 3.3: Analytical solution for the position (blue) and velocity (green) of the undamped spring-block oscillator

subsequent times. One can also get the velocity of the block, $v_x(t) = \frac{dr_x(t)}{dt} = -\omega \; r_{xo} \sin\omega t$.

Plotting the motion of the undamped spring-block system {using K=40, m=10, r_x(t=0)=2,v_x(t=0)=0}, we get Fig 3.3.

Remarks:

(i) *Physical interpretation:* If the block is at rest when the spring is in its relaxed (unstretched and uncompressed) state, then there will be no force due to the spring, and the block will be content to just sit there. However, what we have done is to gently move the block by stretching the spring, and to let go of it. Now, the spring immediately starts to pull back on the block and starts to accelerate the block towards itself. You can view this as the spring trying to get back to its relaxed (unstretched and uncompressed) state. However, when the block returns to the position where the spring does not exhibit any force on it, it has acquired a certain velocity, and so it continues to move on, overshooting the point (r_x=0) where the spring does not exhibit any force on it. That is, the block now starts to compress the spring. The spring of course resists this by now providing a force in the opposite direction, and this slows the block down. However, by the time the block has been brought completely to rest, the spring has become compressed (by r_{xo}, actually!), and so the resulting force from the spring now pushes the block back, and once again overshoot occurs. Thus the spring-block system moves back and forth, back and forth, oscillating forever (assuming no friction and damping).

(ii) The spring-block system is said to oscillate *harmonically*. How long does it take to complete one round trip? Answer: T=2π/ω. This is called the *period* of oscillation. The *frequency* of oscillation (the number of oscillations per unit time) is f=1/T=ω/2π. The unit of frequency is (1/s) more commonly known as Hertz (Hz).

The constant ω is called the *angular frequency* of oscillation. Its units are radians/sec. Note that the period (or frequency of oscillation) depends only on the parameters of the spring-block system (mass and spring constant) and not on the initial conditions.

(iii) Some of you might be uncomfortable with what we have done. You might wonder whether there are any other functions that satisfy the DE(3.6)? And, you might come up with: $r_x(t) = Ce^{\alpha t}$ where C and α are constants. Indeed, if you substitute this in the DE (3.6), you will find that this is a solution to it provided:

$\alpha^2 C.e^{\alpha t} = -\frac{K}{m} C.e^{\alpha t}$. So this is indeed a solution to the DE after all, provided our:

$\alpha = j\sqrt{\frac{K}{m}} = j\omega$ where $j = \sqrt{-1}$ is the familiar imaginary number (you might have used 'i' to denote this, but we need 'i' for later). And you might also realize that: $r_x(t) = De^{-\alpha t}$ is also a valid solution, and even better:

$$r_x(t) = Ce^{j\omega t} + De^{-j\omega t}$$

also satisfies the DE. Now, this is getting interesting. First of all, the position of the block, which is a real thing, apparently is given by something complex or imaginary! Secondly, the above solution to the DE *looks* nothing like the solution involving sines and cosines that we got earlier. The key word in the last sentence is *'looks'*. If you know your calculus and complex variable theory, then you will realize that the two solutions are actually the same! I am not going to prove this to you in general, but let us consider the same initial conditions as before: $r_x(t=0)=r_{xo}$ and $v_x(t=0)=0$. Then, using our new complex exponential solution, we find that $C=-D= r_{xo}/2$, and so:

$r_x(t) = r_{xo}\left[\frac{e^{j\omega t} + e^{-j\omega t}}{2}\right]$. Now, I will leave it as an exercise for you to show that the term in the square brackets is actually a real function (none of this 'j' business) and the real function is just cosωt! [Hint: Recall Taylor Series expansions for cosines, sines and exponential? Or Euler's formula: $e^{j\theta} = \cos\theta + j\sin\theta$.]

(iv) Using trigonometric identities, it can be shown (you will show in PSet 3) that (3.7) can also be written as:

$$r_x(t) = E\sin(\omega t + \phi)$$

where $E = \sqrt{A^2 + B^2}$ is called the **amplitude** of oscillation

and $\phi = \tan^{-1}\frac{B}{A}$ is called the **phase** of the oscillation.

At this point, we have an analytical solution to the spring-block oscillator. Our numerical solution to this problem did not quite work out well, but we will return to that topic shortly and develop a better algorithm that works very well. But, right now we need to look at some physical concepts such as energy and momenta. That will give us a better appreciation of what is going on with the mechanics of the spring-block oscillator.