II Dynamics of Particles

II. 1. Motion of a particle:

In EA2, you encountered the basic concepts of mechanics. You learnt how to solve for the motion of rigid particles under the action of external forces. Let us quickly recapitulate Newton's laws of motion.

First, the **linear momentum p** of a particle is defined as mass m times its velocity **v**:

$\mathbf{p} = m\,\mathbf{v}$.[*]

Then, Newton's laws can be stated as follows.

First Law: The linear momentum of a particle is conserved (does not change) unless the particle is acted upon by external forces. This is equivalent to saying that a particle will remain in its state of rest or uniform motion unless acted upon by external forces.

Second Law: The time rate of change of linear momentum of a particle is equal to the net external force acting on it.

$$\text{That is: } \mathbf{F} = \frac{d\mathbf{p}}{\mathrm{dt}} = \frac{d(m\mathbf{v})}{\mathrm{dt}} = \overbrace{m\frac{d\mathbf{v}}{dt}}^{\text{neglecting relativity}} = m\mathbf{a} \qquad (2.1)$$

A particle acted upon by an external force will therefore accelerate in the direction of the force with a magnitude proportional to the magnitude of the applied load.

Third Law: If a body A exerts an action (force) on a body B, the body B exerts a reaction (force) on body A of equal magnitude and opposite direction.

Remarks:

(i) Recall that Newton's law (2.1) is valid in any reference frame as long as it is inertial (that is it is not accelerating, and this includes rotation).

(ii) We can use (2.1) in Cartesian, polar or any other coordinate system. We just need to remember that we express both the force and the acceleration vectors in the same system.

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[*] Bold letters will be used to denote vectors. (In class, we use under-tilde to denote vectors.}

Cartesian coordinates:

$$
\begin{aligned}
F_x\mathbf{i}+F_y\mathbf{j}+F_z\mathbf{k} &= m\left\{a_x\mathbf{i}+a_y\mathbf{j}+a_z\mathbf{k}\right\}\\
&= m\left\{\frac{d\mathrm{v}_x}{dt}\mathbf{i}+\frac{d\mathrm{v}_y}{dt}\mathbf{j}+\frac{d\mathrm{v}_z}{dt}\mathbf{k}\right\} \qquad (2.2)\\
&= m\left\{\frac{d^2r_x}{dt^2}\mathbf{i}+\frac{d^2r_y}{dt^2}\mathbf{j}+\frac{d^2r_z}{dt^2}\mathbf{k}\right\}
\end{aligned}
$$

where $\mathbf{i},\mathbf{j},\mathbf{k}$ are unit vectors along the x,y and z-directions; $\mathbf{F}$ is the force vector, **a** is the acceleration vector, and **r** is the position vector.

Normal and Tangential components:

$$
\begin{aligned}
F_t\mathbf{e_t}+F_n\mathbf{e_n} &= m\left\{a_t\mathbf{e_t}+a_n\mathbf{e_n}\right\}\\
&= m\left\{\frac{dv}{dt}\mathbf{e_t}+\frac{v^2}{\rho}\mathbf{e_n}\right\} \qquad (2.3)
\end{aligned}
$$

Where $\mathbf{e}_{\mathrm{t}}$ and $\mathbf{e}_{\mathrm{n}}$ are unit vectors tangential and normal to the path, and ρ is the radius of curvature of the path.

(iii) In general, if we know the external forces acting on an object, we can use the above to compute the trajectory of the object. We will work out a few examples later.

II. 2. Motion of Systems of Particles:

It turns out that Newton's law **F**=m**a** holds for a system of particles, and indeed even for bodies of finite size (not just particles) provided we reinterpret its meaning somewhat: ie, if we consider in '**F**' the *net* external forces acting on the *system*, and if 'm' denotes the *total* mass, and '**a**' refers to the acceleration of a point called the **center of mass** of the system of particles. To see this, consider a collection of N particles as our system. Then, let the *ith* particle have a mass m_i, and be located at position ${}_{\mathrm{ri}}$ with respect to some chosen coordinate system. Let us consider the forces on the ith particle in two parts: $\mathbf{f}_{\mathrm{ij}}$ is the force on the ith particle exerted by the jth particle in the system, and $\mathbf{F}_{\mathrm{i}}$ is the force exerted on the ith particle by something external to the system. [For instance, if we consider the earth and the moon to be our system, then the mutual gravitational forces will be included in $\mathbf{f}_{\mathrm{ij}}$ and the pull of the sun and all other external bodies will be lumped in $\mathbf{F}_{\mathrm{i}}$.]

Let us now apply Newton's second law to the ith particle:

$$\sum_j \mathbf{f}_{ij}+\mathbf{F}_i = m_i\frac{d^2\mathbf{r}_i}{dt^2} \qquad (2.4)$$

There are N such equations, one for each particle. Suppose we sum all these N equations together:

$$\sum_i \sum_j \mathbf{f}_{ij} + \sum_i \mathbf{F}_i = \sum_i m_i \frac{d^2\mathbf{r}_i}{dt^2} \tag{2.5}$$

Let us look at the above carefully. What is the first term? It is the sum of all forces exerted on one another by the particles within the system. But by Newton's third law, if particle i exerts a force $\mathbf{f}_{ij}$ on the jth particle, then particle j exerts an equal but opposite force $\mathbf{f}_{ji}$ on the jth particle: ie $\mathbf{f}_{ji} = -\mathbf{f}_{ij}$. So when you sum all these internal forces up, they must add to zero! So we are left with sum of all external forces on the left side and we would like to say that this is equal to the acceleration of sum point. To do this, let us define the *center of mass* of the particles as that position **r** given by:

$$\mathbf{r} = \frac{\sum_i m_i \mathbf{r}_i}{\sum_i m_i}. \tag{2.6}$$

Then, the right side becomes precisely what we want it to be!

$$\sum_i \mathbf{F}_i = m\frac{d^2\mathbf{r}}{dt^2} \text{ where m is the } \textit{total} \text{ mass of the system.}$$

Remarks:

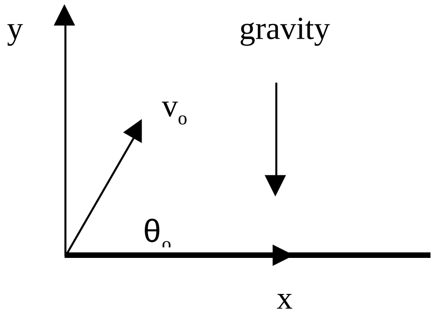
(i) Above applies to gases, liquids, and solid bodies since all of them can be conceptually thought of as being composed of several particles.

(i) On the meaning of the center of mass: The center of mass is kind of an "average" position of all the particles, where the average is computed using mass as a weighting factor. You will also encounter creatures such as centers of gravity, pressure, population etc, but they are all basically of the same form (2.6) but using the relevant weighting factors (weight, pressure, population density etc). Note that the center of mass of a body need not be a point inside the body. Example: vinyl records, which have holes in the middle.

Example 1. (Projectile motion with no drag):

Consider the near-earth motion of a body that is acted upon by earth's gravity. We will neglect air drag and any other forces on the body. For near-earth motion, the earth can indeed be treated as flat, and so we locate an xy-coordinate frame attached to earth.

Mass of projectile: m
Let v_o be the intial speed of the projectile thrown at an angle θ_o.
Acceleration due to gravity: g



The force on the projectile is:

$\mathbf{F} = -mg\mathbf{j}$ and it is constant throughout the motion of the body.

The equations of motion for the projectile are therefore:

$$\frac{d\mathbf{v}}{dt} = \mathbf{a} = \frac{1}{m}\sum \mathbf{F} = -g\mathbf{j}$$

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}$$

The initial conditions are:

$$\mathbf{r}(t=0) = 0\mathbf{i} + 0\,\mathbf{j}$$

$$\mathbf{v}(t=0) = \left(v_o \cos\theta_0\right)\mathbf{i} + \left(v_o \sin\theta_0\right)\mathbf{j}$$

Integrating the equations of motion, and using the initial conditions, we find:

$$\mathbf{v}(t) = \left(v_o \cos\theta_o\right)\mathbf{i} + \left(v_o \sin\theta_o - gt\right)\mathbf{j}$$

$$\mathbf{r}(t) = \left(v_o t \cos\theta_o\right)\mathbf{i} + \left(v_o t \sin\theta_o - \left(gt^2/2\right)\right)\mathbf{j}$$

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You can look up Bedford and Fowler for several additional problems that you might want to solve. You have done a lot of these in EA2, and you should review that material now. Let us now move on to two problems that we can set up but we may not be able to solve readily.

Example 2. The projectile problem with air drag:

Let us re-consider the projectile problem, but now let us include air resistance (drag) which has experimentally been found to be proportional to the square of the speed of the projectile and which acts in a direction opposing the motion.
The force on the body is due to weight acting in the negative y-direction and drag acting tangential to the path of the particle and opposing the motion (that is in a direction that is opposite to the velocity vector.

$$\sum \mathbf{F} = -mg\mathrm{j} - C|\mathbf{v}|^2 \underbrace{\left[\frac{\mathbf{v}}{|\mathbf{v}|}\right]}_{unit\ \ vector} = \underbrace{\left\{-C\sqrt{v_x^2+v_y^2}\,v_x\right\}}_{\mathrm{F_x}}\mathbf{i} + \underbrace{\left\{-mg - C\sqrt{v_x^2+v_y^2}\,v_y\right\}}_{\mathrm{F_y}}\mathbf{j}$$

The resulting equations of motion are:

$$\frac{d\mathbf{v}}{dt} = \sum \mathbf{F} = -mg\mathrm{j} - C|\mathbf{v}|^2 \underbrace{\left[\frac{\mathbf{v}}{|\mathbf{v}|}\right]}_{unit\ \ vector} = \underbrace{\left\{-C\sqrt{v_x^2+v_y^2}\,v_x\right\}}_{\mathrm{F_x}}\mathbf{i} + \underbrace{\left\{-mg - C\sqrt{v_x^2+v_y^2}\,v_y\right\}}_{\mathrm{F_y}}\mathbf{j}$$

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}$$

Remarks: If we try to integrate the above we will be stumped. We find that we cannot analytically integrate the first of these equations because the creature we want to solve for (**v**) appears on the right side as well. This is a full-fledged differential equation (and a non-linear one at that because the right side involves the square of **v**) and it is not possible for us to solve this analytically. So what do we do? Hold on…

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Example 3. Motion of an Electron in an Electromagnetic Field:

The force on a charged particle in an electromagnetic field is given by:

$$\mathbf{F} = q\left\{\mathbf{E} + \mathbf{v}\times\mathbf{B}\right\}$$

where q is the charge on the particle, **E** is the electric field, and **B** is called the magnetic flux density, and **v** is the velocity of the particle. Once again, if we write the equation of motion for this we find that it is not easy to integrate them if the magnetic field exists. (Try it.)

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II.3. Numerical Integration of the Equations of Motion:

In the last two examples of the previous section, we found that we could not easily integrate the equations of motion analytically. This was because the force on the body was affected by the motion itself. In such cases, we seek *numerical* solutions. Our goal now is to numerically solve for the motion of a particle given its initial state (position, velocity) and subject to external forces (which may be a function of time, position, velocity). Consider one-dimensional motion first (say along the x-direction). Then, we are given:

mass of the particle m
initial position at time t_o is $r_x(t_o)$
initial velocity at time t_o is $v_x(t_o)$.
Net external force on the body is $\sum F_x(t, r_x, v_x)$

We need to figure out the subsequent motion of the particle. We start with Newton's second law, $F_x = ma_x$, which we write as:

$$\frac{dv_x}{dt} = \frac{1}{m}\sum F_x(t, r_x, v_x) \tag{2.7}$$

If the functional dependence of the force on time, position or velocity were simple, we can try to analytically integrate the above. In general, however, it is rather hard if not impossible to do so. We therefore look for an *approximate* numerical solution to the above by considering a small increment of time Δt from time t_o. From the definition of the derivative, we have:

$$\frac{dv_x}{dt}(t_o) = \lim_{\Delta t \to 0} \frac{v_x(t_o + \Delta t) - v_x(t_o)}{\Delta t} \approx \underbrace{\frac{v_x(t_o + \Delta t) - v_x(t_o)}{\Delta t}}_{\text{if } \Delta t \text{ is small}} \tag{2.8}$$

Using (2.7) in (2.8) we have:

$$\frac{v_x(t_o + \Delta t) - v_x(t_o)}{\Delta t} \approx \frac{1}{m}\sum F_x(t_o, r_x(t_o), v_x(t_o)) \tag{2.9}$$

Therefore, the velocity at time $t_1 = t_o + \Delta t$ is approximately:

$$v_x(t_o + \Delta t) \approx v_x(t_o) + \frac{1}{m}\sum F_x(t_o, r_x(t_o), v_x(t_o))\,\Delta t \tag{2.10}$$

where the approximation becomes better and better as Δt becomes smaller and smaller.

We can get the new position of the particle in like manner starting with:

$$\frac{dr_x}{dt}(t_o) \approx \lim_{\Delta t \to 0} \frac{r_x(t_o + \Delta t) - r_x(t_o)}{\Delta t} = v_x(t_o) \tag{2.11}$$

which yields:

$$r_x(t_o + \Delta t) \approx r_x(t_o) + v_x(t_o)\Delta t \tag{2.12}$$

So, we now know the approximate new position and velocity of the particle at time $t_1 = (t_o + \Delta t)$. We can use these new values at time t_1 and repeat the process for the next time step: $t_2 = (t_1 + \Delta t)$ and so on.

Remarks:

(i) This is called the forward Euler method of integration. Forward because we are using the current values of velocity (or acceleration) to calculate the next values of position (or velocity) etc.

(ii) The accuracy of the solution depends on our time steps Δt - if we take too large ("coarse") a time step, the approximation of the derviative becomes poorer and the solution will be less accurate.

II.3.1 ***Matlab coding of the forward Euler algorithm*****:** Let us code the forward Euler algorithm in MATLAB. Simultaneously, we will extend the above idea to motions in two or three dimensions.

Input:
mass of the particle m
initial time t_o =➔ [time]
initial position at time t_o is $\mathbf{r}(t_o) = r_x(t_o)\mathbf{i} + r_y(t_o)\mathbf{j} + r_z(t_o)\mathbf{k}$ =➔ [xold,yold,zold]
initial velocity at time t_o is $\mathbf{v}_x(t_o) = v_{xo}\,\mathbf{i} + v_{yo}\,\mathbf{j} + v_{zo}\,\mathbf{k}$=➔ [vxold,vyold,vzold]
external force on the body is $\sum \mathbf{F}(t,\mathbf{r},\mathbf{v}) = \sum F_x(t,\mathbf{r},\mathbf{v})\mathbf{i} + \sum F_y(t,\mathbf{r},\mathbf{v})\mathbf{j} + \sum F_z(t,\mathbf{r},\mathbf{v})\mathbf{k}$
(given as a function or table) [fx,fy,fz]

Want to find position **r**(t), velocity **v**(t), acceleration **a**(t), force **F**(t) et cetera for subsequent times.

Input Δt =➔ time_step

(*) **Compute** the approximate positions at the next time step:
$r_x(t_o + \Delta t) \approx r_x(t_o) + v_x(t_o)\Delta t$ =➔ xnew = xold + vxold*time_step
$r_y(t_o + \Delta t) \approx r_y(t_o) + v_y(t_o)\Delta t$ =➔ ynew = yold + vyold*time_step
$r_z(t_o + \Delta t) \approx r_z(t_o) + v_z(t_o)\Delta t$ =➔ znew = zold + vzold*time_step

Compute forces: [fx, fy, fz] using current time, position r(xyz)old, velocity v(xyz)old

Compute the approximate velocities at the next time step:
$v_x(t_o + \Delta t) \approx v_x(t_o) + \frac{1}{m}\sum F_x(t_o, \mathbf{r}(t_o), \mathbf{v}(t_o))\,\Delta t$ =➔ vxnew = vxold+(fx/m)*time_step

$v_y(t_o + \Delta t) \approx v_y(t_o) + \frac{1}{m} \sum F_y(t_o, \mathbf{r}(t_o), \mathbf{v}(t_o)) \, \Delta t$ =➔ vynew = vyold+(fy/m)*time_step

$v_{yz}(t_o + \Delta t) \approx v_z(t_o) + \frac{1}{m} \sum F_z(t_o, \mathbf{r}(t_o), \mathbf{v}(t_o)) \, \Delta t$ =➔ vznew = vzold+(fz/m)*time_step

Swap new to old and repeat from (*)