VI.4 DAMPED OSCILLATIONS

Consider the {spring-damper}-mass system shown.

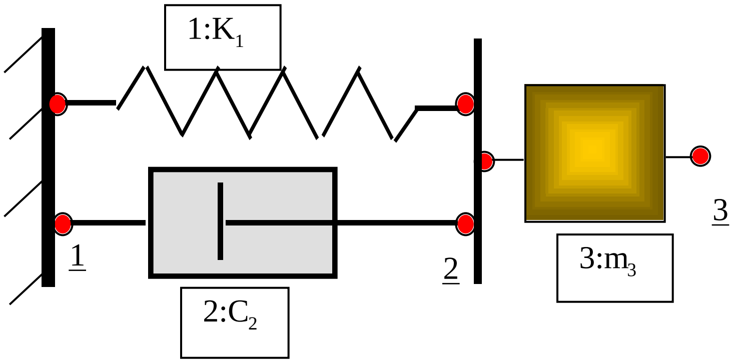


Figure 6.11: Damped spring-mass system

There is a slight complication to this system. Is this one-dimensional? Actually, in this case, we have the spring and the damper in parallel (whereas thus far the elements were all connected in series). However the system is still one-dimensional in that the *ties* (the big vertical lines connecting the nodes of the elements in parallel) are assumed not to rotate, but only to translate in the x-direction. Therefore the entire tie connector can be thought of as just an extended node, and this is how we will treat them.
Let us now crank our analysis machinery.

State Variables: $\underline{X} = \left\{ \begin{matrix} r_{sp1} \\ v_{m3} \end{matrix} \right\}$

Geometric Relations:

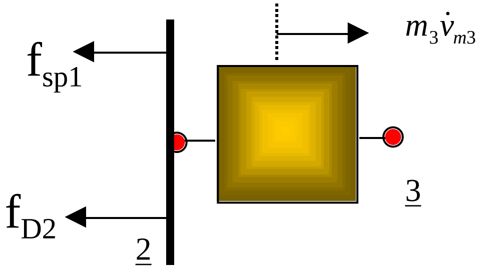
$$(i) \qquad x_1 = 0$$

$$(ii) \qquad r_{sp1} = x_2 - x_1 = x_2$$

$$(iii) \qquad r_{D2} = x_2 - x_1 = x_2 \rightarrow v_{D2} = \dot{x}_2$$

$$(iv) \qquad x_3 - x_2 = \text{constant} \rightarrow \dot{x}_3 = \dot{x}_2 = v_{m3}$$

Equilibrium Relations:



$$(v) \qquad -f_{sp1} - f_{D2} = m_3\dot{v}_{m3}$$

Constitutive Relations:

$$(vi) \qquad f_{sp1} = K_1 r_{sp1}$$

$$(vii) \qquad f_{D2} = C_2 v_{D2}$$

State Equations: (by now you should be able to do this)

$$\boxed{\begin{Bmatrix} \dot{r}_{sp1} \\ \dot{v}_{m3} \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{K_1}{m_3} & -\dfrac{C_2}{m_3} \end{bmatrix} \begin{Bmatrix} r_{sp1} \\ v_{m3} \end{Bmatrix}} \qquad (\dagger)$$

Feeding this to an appropriate MATLAB m-file with K1=20, m3=10, C2=2, and initial condition X(0)={0, 0.2}', we find:

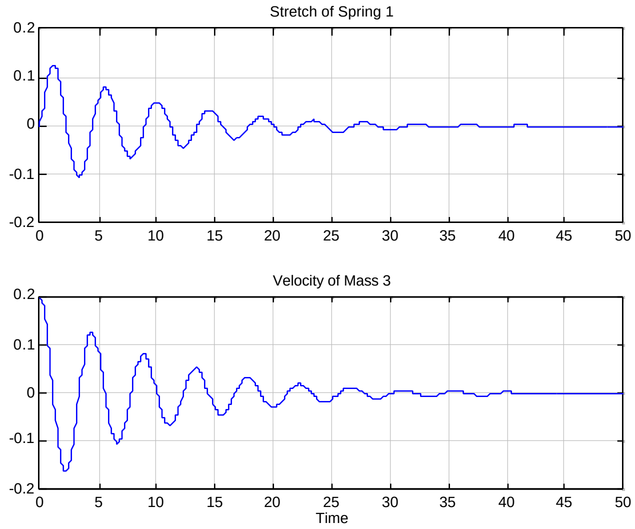


Figure 6.13: Damped oscillation of a spring-damper system

Note that this system exhibits some oscillatory behavior (like the spring-mass oscillator), but because of the energy dissipating damper, the amplitude of oscillations decays down with time (compare with the result of the spring-damper system which exhibited an exponential behavior). We were able to deduce the analytical solution to the spring-damper and the spring-mass systems. Now, try to guess the form of the analytical solution for the spring-damper-mass system above before reading on.

Analytical solution: Just as we did for the undamped spring-mass oscillator of Chapter 3, I am going to swap out one of the state variables (v_{m3}) in favor of the stretch of the spring (r_{sp1}). Note that for this case, the stretch of the spring is the same as that of the position of the mass. This requires some simple algebra.

The second of our state-equations (†) reads:

$$\dot{v}_{m3} = -\frac{K_1}{m_3} r_{sp1} - \frac{C_2}{m_3} v_{m3} = -\frac{K_1}{m_3} r_{sp1} - \frac{C_2}{m_3} \dot{r}_{sp1}$$

But:

$$v_{m3} \overset{(iv)}{\overbrace{=}} \dot{x}_2 \overset{(ii)}{\overbrace{=}} \dot{r}_{sp1} \rightarrow \dot{v}_{m3} = \ddot{r}_{sp1}$$

and therefore, eliminating any mention of v_{m3}, we get:

$$\ddot{r}_{sp1} + \frac{C_2}{m_3} \dot{r}_{sp1} + \frac{K_1}{m_3} r_{sp1} = 0$$

which in full notation looks like:

$$\frac{d^2 r_{sp1}}{dt^2} + \xi_D \frac{dr_{sp1}}{dt} + \xi_S r_{sp1} = 0, \qquad (\dagger\dagger)$$

where we let $\xi_D = \frac{C_2}{m_3}$; $\xi_S = \frac{K_1}{m_3} \equiv \omega_n^2$ for convenience.

In the absence of a damper, (††) becomes the familiar spring-mass oscillator equation, and the frequency of oscillation of such an undamped system will just be ω_n , where I am now using a subscript ‘n’ to remind me that it is the frequency of oscillation of the undamped system (not the actual damped system I am considering now).

What we have in (††) is a *second-order differential equation* for r_{sp1}, and it is quite similar to the one we obtained for the spring-mass oscillator, except that there is one additional term here involving the damper. In fact, these DEs fall under the class of what are called linear differential equations with constant coefficients (linear because we do not have powers or some other complicated forms of r_{sp1} ; and constant coefficients because all the derivative terms are multiplied only by constants and not functions of time.)

Earlier we guessed the solution to be sines or cosines which turned out be the same as complex exponentials, and now I am going to tell you that linear DEs with constant coefficients will always have solutions that are exponential.[#] That is, we will always *seek* a solution of the form:

$$r_{sp1} = e^{\alpha t}$$

Substituting above in (††), we find that only specific α's will satisfy (††):

$$\alpha^2 + \xi_D\alpha + \xi_S = 0$$

The roots of this equation are:

$$\alpha_{1,2} = -\frac{\xi_D}{2} \pm \sqrt{\left(\frac{\xi_D}{2}\right)^2 - \xi_S}$$

and therefore both:

$$r_{sp1}(t) = e^{\left\{-\frac{\xi_D}{2} + \sqrt{\left(\frac{\xi_D}{2}\right)^2 - \xi_S}\right\}t} \text{ and } r_{sp1}(t) = e^{\left\{-\frac{\xi_D}{2} - \sqrt{\left(\frac{\xi_D}{2}\right)^2 - \xi_S}\right\}t} \text{ will satisfy the DE.}$$

In fact, any linear combination of these two solutions will also be a solution of (††). That is, the general solution to (††) is:

$$r_{sp1}(t) = Ae^{\left\{-\frac{\xi_D}{2} + \sqrt{\left(\frac{\xi_D}{2}\right)^2 - \xi_S}\right\}t} + Be^{\left\{-\frac{\xi_D}{2} - \sqrt{\left(\frac{\xi_D}{2}\right)^2 - \xi_S}\right\}t} \qquad (¥)$$

where A and B are constants to be determined from initial conditions. One way to check if we have got the right solution is to put the solution (¥) back into the DE (††) and see if it is satisfied. You can either do that now, or save yourself some trouble, and take my word for it that it does.

To check the behavior of the solution, let me first rewrite (¥):

$$r_{sp1}(t) = e^{-\frac{\xi_D}{2}t}\left\{Ae^{\left\{+\sqrt{\left(\frac{\xi_D}{2}\right)^2 - \xi_S}\right\}t} + Be^{\left\{-\sqrt{\left(\frac{\xi_D}{2}\right)^2 - \xi_S}\right\}t}\right\} \qquad (¥¥)$$

The first factor is just an exponentially decreasing function with time, and indeed this is the reason why we saw our numerical solution die away. To figure out the behavior of the term in curly brackets, we need to consider three cases:

[#] Please see the associated web notes on Euler's Differential Equations.

Case (a): Overdamped case: Here the term in the square root is positive, ie: $\xi_D^2 > 4\xi_S$. Then, clearly we get real numbers for the arguments of the exponential functions inside the curly brackets as well. It is possible depending on the parameters that the solution first grows (there is one growing exponential function inside the curly brackets), but soon the strong exponential decay factor sitting in front takes over. Therefore the overall solution is just some kind of decay. That is, the damper is so “strong” that the system does not oscillate for the overdamped case, but just goes exponentially to zero spring stretch. Such behavior is shown in Fig. 6.14 (left) for the system parameters indicated.

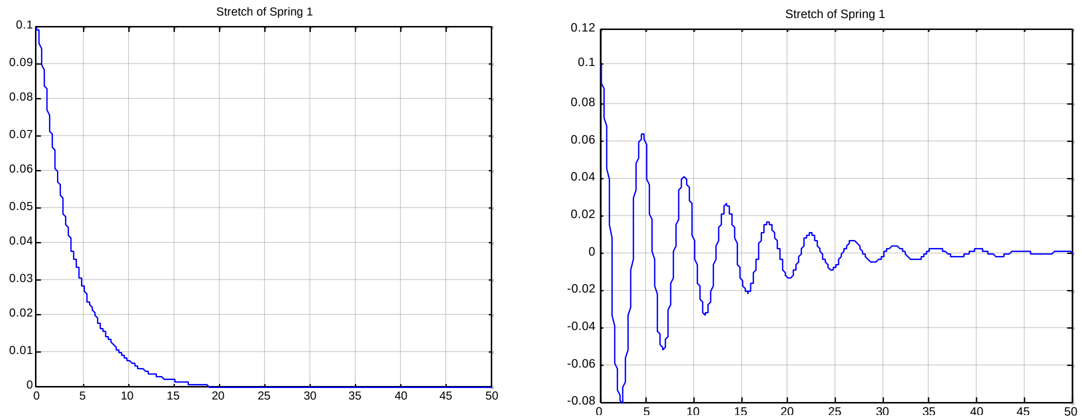


Figure 6.14: Left: Overdamped case {K1=20, C2=80, m3=10)
Right: Underdamped case {K1=20, C2=2, m3=10)

Case (b) Underdamped case: Here the term in the square root is negative, ie: $\xi_D^2 < 4\xi_S$. This means that the argument of the exponentials inside the curly brackets become complex! But we have seen such a thing before for the undamped spring-oscillator problem. There I argued that one could use Taylor series expansions for complex exponentials and show that they are related to oscillatory things: *sines* and *cosines.* This holds true here too, of course, and actually for the underdamped case, we can re-cast (¥¥) as:

$$r_{sp1}(t) = e^{-\frac{\xi_D}{2}t}\left\{C\sin\left(\omega_D t\right) + D\cos\left(\omega_D t\right)\right\}$$

where $\omega_D = \sqrt{\omega_n^2 - \left(\frac{\xi_D}{2}\right)^2}$,

and C and D are constants (related to our previous A and B) that can be obtained from initial conditions. What this says is that for the underdamped case, we have some kind of an oscillatory response, but the amplitude of the oscillations decays exponentially due to the multiplicative exponential decay term sitting in front of the curly brackets. Note also that the frequency of oscillations is not the same as that for an undamped spring-mass system, but rather is reduced to ω_D due to the presence of the damper. Fig. 6.14 (right) shows such damped oscillatory behavior for the system parameters indicated.

Case (c) : Critically damped case: Here the term in the square root is just zero. One can analytically explore this borderline behavior as well, but there is a slight complication here that is not worth getting into right now.

The physics vs the math of things:

Physically we saw that for an ideal undamped spring-mass system, the potential energy in the spring is transferred to the mass as kinetic energy back and forth. Since there is no energy loss in our ideal system, the oscillatory behavior of the system goes on forever with the amplitude of oscillations never changing. We think of this as the spring and the mass tossing their potential and kinetic energy to each other back and forth. But how did the system get its energy to start with? Something external to the system must have done work (in setting the system off at its initial state for instance). We have seen how we might do this by initially stretching or compressing the spring, or by initially imparting some kinetic energy to the mass. However, once the ideal spring-mass system got going, we found that it keeps on oscillating. In reality, of course, friction and damping are present to steal some energy away. That is why for our damped spring-mass system, the system eventually comes to a halt. If the energy stealing ('dissipating') device in the system is sufficiently "strong" ($\xi_D^2 > 4\xi_S$) the energy input to the system initially (either to the spring as potential energy or to the mass as kinetic energy) is completely lost by the damper without causing oscillatory motion in the system. If the energy stealing device in the system is not strong enough ($\xi_D^2 < 4\xi_S$), the spring and the mass do toss their potential and kinetic energy back and forth, but all the time the damper steals some of the mechanical energy (and converts it to heat), and so the oscillations slowly decrease till the system eventually comes to a halt.

We should be happy that our physical picture agrees with our mathematical picture. Turns out that there are other systems in other domains which have energy storing devices (akin to springs), energy dissipating devices (akin to dampers), and inertial devices (akin to masses), where very similar behavior occurs. We will be able to use our physics intuition (in terms of energy considerations) and the mathematical machinery (in terms of solutions to differential equations) to these other systems as and when we encounter them.