VI SYSTEMS DYNAMICS – MECHANICAL SYSTEMS

VI.1 Modeling Mechanical Systems:
We have already looked at some simple one-dimensional mechanical systems involving springs and blocks:

![Diagram of a spring-block oscillator](image)

**Frictionless, No gravity**

(a)

![Diagram of a spring-block-spring device](image)

**Frictionless; no gravity**

(b)

**Figure 6.1:** (a) The undamped spring-block oscillator of Chapter III; and (b) a spring-block-spring device.

In our analyses of these systems we have assumed that the block is rigid and has mass, and the spring is massless but is deformable. This is what we called a lumped parameter idealization where all the “springiness” (deformability) is attributed to the spring and all the “massiveness” is attributed to the block. In reality, real springs do have mass, and real blocks are deformable. However, ours is a useful approximation (if, for instance, the mass of the block is a lot more than that of the spring, and the stiffness of the block is a lot

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† The course material from this point onwards draws heavily from the EA3 hyperbook of Prof. Peshkin. I will make available selected links to that hyperbook (available on the web only) at a later date. Some of
higher than that of the spring). Also, we can actually model more complex useful real life systems with such simple idealized elements. For instance, the spring-block system might represent a heavily-loaded table or a unicycle (view Fig. 6.1a rotated CCW 90°). Of course more complex mechanical systems might have to be modeled with several elements. The system might have elements that not only “store” energy (potential energy in springs) and have inertia (blocks), but there may also be elements that “lose” energy (dampers) which we shall see shortly. And a real system will more often than not be two- or three-dimensional, and may have components that rotate, and not just translate in one-dimension. For simplicity, however, we will now consider mechanical systems that can only translate (no rotation) in one-dimension. Figure 6.2 shows one such system with several springs and blocks. This might represent a part of a high-rise building where the springs might represent the supporting columns, and the blocks represent the mass of the floors, all appropriately “lumped” at the locations shown.

![Figure 6.2: A 12mass-13-spring system (courtesy: EA3 hyperbook)](image)

In trying to look at the dynamics of such systems, it is convenient to use a formal systems dynamics approach. In this approach, we consider the system to be comprised of several elements each with one defining property. These elements are connected to one another only at their end points, which we shall call nodes or connections. In EA2 you used a similar approach (recall the finite-element analyses of trusses?), but there you looked only at static problems (and so the mass of the various elements played no significant role in the analysis as inertia did not enter the picture). In fact, the only kind of elements you saw there were truss and spring elements, which only had springiness (elastic deformability) associated with them. We will now look at three elements for one-dimensional translational mechanical systems that are important in modeling their dynamics.
VI.2: Elements for one-D translational mechanical systems
There are three common elements here of which you have already seen two: (i) springs, (ii) masses, and (iii) dampers which are elements that provide a means of energy dissipation.

VI.2.1 Spring Elements: The defining characteristic of a spring element is that it relates the force acting through it to the relative displacement (stretch) of its two ends from its unstretched state. Springs are assumed massless. Note that I am talking about forces through a spring. Since we have assumed that springs are massless, if we draw the free-body diagram of a spring, Newton’s second law (note m=0): \( \sum F = 0 \) indicates that the forces on either end of the spring must be of equal magnitude and they act opposing each other as shown.

\[ f_{sp} \]

\[ f_{sp} \]

**Figure 6.2:** Spring in tension (top) and compression (bottom).

Notation: We will use the symbols: \( f_{sp} \) for the force through a spring, and \( r_{sp} \) for the relative displacement (stretch) of the spring.

Convention: The force through a spring (henceforth: spring force) is taken as positive if it is tensile (spring stretches and therefore \( r_{sp} > 0 \)) and negative if it is compressive (spring compresses and therefore \( r_{sp} < 0 \)). This convention should be familiar to you from EA2.

Terminology: Spring force is called a “through” quantity as it is the same on either side and so can be thought of as being transmitted unchanged through the spring. The stretch of the spring, which is the relative displacement across the two ends of the spring, on the other hand, is called an “across” quantity.

Constitutive relation for springs: The actual relation between the force through a spring and its stretch is called its constitutive relation. Consider the coil spring shown in Fig. 6.3a. If you were to apply a force through the spring and measure the corresponding stretch (in tension Fig. 6.3b and in compression Fig. 6.3c), you might find that the force through the spring is related to its stretch as shown in Fig. 6.3d). We note that the spring force-stretch relation is linear for small amounts of stretch (or compression):
\[ f_{sp} = K r_{sp} \]  

where \( K \) is the spring constant (dimensions: force/length). For larger amounts of stretch, the relation usually becomes highly non-linear (i.e. not proportional) as shown in Fig. 6.3d. If we restrict our attention to only the linear region, the linear spring response is characterized by only one parameter: the spring constant. In this course, we will restrict attention only to the linear response region. But can you figure out why the spring might be expected to behave the way it does for large tension and large compression? [Hint: look at Figs. 6.3b,c and imagine what is happening to the spring]. Note also that in the non-linear response region, springs may no longer be energy storing devices. Why?

(a) Coil Spring

(b) Spring in tension

(c) Spring in compression

(d) Constitutive relation

**Figure 6.3:** Spring elements  (courtesy EA3 hyperbook)

**Remarks:** There are more complicated spring elements in two and three dimensions. All of them have this in common: they relate a load acting through them to their deformation. Examples of such springs: leaf springs, torsional springs etc.
VI.2.2 Mass elements: The defining property of a mass element is its *mass*. These are what we have been calling “blocks” up to now, but they are more commonly just called “mass” and so henceforth we shall also do so. Masses are assumed to be rigid. So the only parameter characterizing mass elements is their mass $m$.

Analogous to spring elements, we will pictorially represent a mass element as a block with two *rigid* handles which may be connected to other elements at the (red) nodes.

![Figure 6.4: Mass element.](image)

Note that we cannot say that the forces acting across the mass element are equal in general. From Newton’s laws, all we can say is that:

$$ f_{\text{right}} - f_{\text{left}} = m\dot{v}_m $$  

where $v_m$ is the absolute velocity of the mass, which is the same as the absolute velocity of the nodes that the *rigid* mass is attached to via the rigid handles. Note that since the mass is rigid, the two nodes attached to the mass must move together identically.

*Constitutive laws for mass elements?* Note that we can think of mass elements as creatures that relate the *net force* acting on them to their absolute *acceleration* (with respect to some inertial frame of reference). This is analogous to springs where the spring force was related to the spring stretch (a force-*displacement* relation). For masses, Newton’s second law (a force-*acceleration* relation) therefore plays the role of a “constitutive” law. This analogy should not be carried too far in that springs come in a variety of ways with different constitutive laws (I do not mean just different spring-constants for a linear spring, but different *functional* forms for the spring force-stretch relations such as a cubic behavior: $f_{sp} = k_1 r_{sp} + k_2 r_{sp}^3$ etc). Thus constitutive relations for springs are hardly universally valid. Newton’s laws however are not merely a “constitutive” relation, but rather universal laws that are always valid (within the assumptions of classical mechanics). That is, don’t go looking for mass elements that behave like: $f_{\text{right}} - f_{\text{left}} = m_1\dot{v}_m + m_2 v_m^{\text{whatever}}$ or what not!
VI.2.3 Damper elements: This last element is a new one for us. Our two previous elements related a force to a relative displacement (springs), and a force to an acceleration (masses). Are there elements that exhibit a force-velocity relation? Turns out that there are devices called dampers that can be made to exhibit such a behavior. A cylinder filled with a viscous fluid through which a piston is made to move is one such simple device that I am pretty sure you must have seen. Dampers will be pictorially represented as:

![Figure 6.5: Damper element](image)

Dampers are assumed to be massless, and so once again, we realize that the force acting on either end of the damper must be of equal magnitude and in opposite direction. We can therefore once again talk of the force acting through a damper, and we will retain the same convention as that for springs. A tensile force through a damper will cause it to extend and a compressive force will cause it to shorten in length. The constitutive law for dampers however differs from that for springs in that the force through the damper is related not to its stretch, but rather to the rate of stretch (or compression), i.e. to the relative velocities across its two ends. This relationship can be complex as shown for a syringe in Fig. 6.6a,b. Often, dampers are designed to have at least an approximately linear constitutive relation (see Fig. 6.6c):

\[ f_D = C v_D \]  \hspace{1cm} (6.3)

where \( f_D \) is the force through the damper (henceforth: damper force); \( C \) is called the damping coefficient, and \( v_D = v_{right} - v_{left} \) is the relative velocity across the two ends of the damper.

(Aside: The reason why linear springs and linear dampers are sought after is because the behavior of systems of such elements is relatively easy to understand and model. Nature, however, might not always oblige us by providing linear responses. Some rather

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\(^†\) Superposed dot is shorthand for time derivative.
fascinating dynamics are observed in systems that include elements exhibiting non-linear behavior. However, we must first learn to crawl before we can walk, and so in this course we will restrict attention to linear elements only.)

Remarks on spring and damper elements:
1. It is to be noted that the damper force is related to the relative rate of lengthening or shortening of the damper (we will simply call this the damper velocity – this is not the absolute velocity of the entire damper, but rather the relative velocity of its two ends; that is damper velocity is an “across” quantity whereas damper force is a “through” quantity).
2. If the damper force is suddenly removed, apparently from (6.3) it instantaneously stops lengthening or shortening. Note that instantaneous stopping means infinite acceleration!

Figure 6.6: (a) Syringe, (b) its constitutive response, and (c) a desired linear damper. (Courtesy EA3 hyperbook)
This somewhat surprising result is actually a consequence of our assumption that dampers are massless, and such behavior cannot quite happen in reality. But we can create a “good” damper that approaches our ideal behavior if we use components which do not have much mass. The same goes for springs where if an external force is suddenly removed, the spring, according to (6.1) should instantaneously return to its unstretched length! Such quirky behavior is not observed in reality because real springs and dampers are made of components which do have mass. Therefore treat our elements as idealizations only, and use care in modeling real systems with such idealized elements.

3. Note that if there is a force through a damper it will keep lengthening (or shortening) unlike a spring which will lengthen (or shorten) up to a point and then stop. Of course, real dampers will also not continue to lengthen or shorten forever because they will either come completely apart or their internal components will lock, ie they will no longer behave as dampers. We will assume that all the mechanical systems we will see in this course are designed by good engineers who do not tax their elements to such extremes.

4. We have previously seen that elastic springs provide conservative spring forces, and therefore we could associate a potential energy. The work done by external forces in compressing or stretching the spring could be thought of as being stored as potential energy in the spring which could be recovered in that it could be used in turn to do work on an external body. Recall the undamped spring-block (“spring-mass” according to our new terminology) oscillator of Chapter 3 where the kinetic energy of the mass was converted to the potential energy of the spring back and forth.

Now, is the damper force conservative? It is easy to see that it is not. Imagine that a damper lengthens by an amount $\delta$ in time $\Delta t$ under the action of a constant damper force $f_D$. The work done by the damper force is $U = f_D \delta = C v_D \delta^2 / (\Delta t)$. This is not just dependent on the end positions (given by $\delta$), but rather depends on how we get there. I could have used a very small damper force and it would have taken a long time for the damper to lengthen to $\delta$ in which case the work done would be very small, or I could use a very large damper force and get the damper to lengthen very fast, in which case the work done would be larger than before. That is, the work done to get the damper from one state to another depends on not just the end states, but how we get to these states. Therefore, the damper force is not a conservative force, and we cannot associate a potential energy with it. Indeed, since I can get the damper to lengthen with
very little work done (use a small force and wait a long time), it is not conceivable that the work done is somehow stored as potential energy to be recovered. Actually, it turns out that the work done on dampers is mostly lost as heat. Therefore dampers are energy dissipative elements, because they lose mechanical energy. We can therefore expect that if we include a damper element to the spring-oscillator system of chapter 3, the oscillations will probably die out in time. We will see shortly that this is exactly what happens.

VI.3 Systems of springs-dampers-masses

We will now develop a systematic procedure to investigate the dynamics of complex one-dimensional mechanical systems.

VI.3.1 A Spring-Mass System: Of course it is always wise to start with a simple example that we have seen before: a 1 spring-1 mass system

![Figure 6.7: A spring-mass system](image1)

Note that I will no longer draw frictionless floors or tracks but these are assumed, and unless otherwise stated, I will be turning off gravity as well. The above spring-mass system will exhibit dynamic behavior only if I do something to it, such as apply an external force to it, or impart a velocity to the block or initially stretch the spring from its relaxed state.

![Figure 6.8: Annotating the system](image2)
Let us first “annotate” the given system. This is quite similar to what you did for finite element meshes in EA2. The first step is to number all the elements in sequence, and write down symbols for their defining parameter such as spring constant or mass. Here $1:K_1$ means my first element is a spring of spring constant $K_1$, and $2:m_2$ means that the second element of my system is a mass of mass $m_2$. I will denote element numbers inside a rectangular box just so I don’t confuse them with node numbers. You can adopt any convenient shorthand that you like as long as it leads to no ambiguity or confusion. Next we also number all the connecting nodes in sequence. I will indicate node numbers with an underbar. For the system under consideration there are two elements and three nodes. Node 2 is what connects the spring to the mass. You might wonder what nodes 1 and 3 are doing. Actually here node 1 is connected to a rigid stationary object such as a wall or ground. Sometimes our system might be anchored to a shaker table (imagine the ground during an earthquake) which can impart a velocity to that node. Such external things that dictate what the velocity of any part of our system must be at any time are called velocity sources, and we will come back to these later. For now, node 1 here is anchored to a rigid immobile ground or wall. And node 3 is just not connected to anything at all. What this says is that the other end of the mass is not connected to anything in this case. The reason we still indicate the node here is because it is possible to have some other element attached to this node in a more complex system. Or else we might have an externally applied force acting on this node (that is, something external might pull or push the mass at this end). We shall call such external forces acting at nodes force sources. Again, we will come to these velocity and force sources later. For now node 3 is connected to nothing at all and no external forces act on it.

In our earlier analysis of the spring-block oscillator, I basically looked at the free-body diagram of the block when the spring was stretched, and then proceeded to write down the equations of motion of the block itself. My state variables there were the position and velocity of the mass. Here, however, I am going to use the stretch of the spring and the velocity of the mass as the state variables. You should be able to quickly figure out that the stretch of the spring is indeed related to the position of the block, and therefore we are not really doing something illegal. Turns out that for complex systems, it is always wise to choose the stretches of springs and the velocities of masses as the state variables. Since the various elements are interconnected, there are always some geometric relations that must hold between the various stretches and various velocities of our elements. Otherwise the elements of our system will have to come apart (for instance, it is not possible for the mass to be moving to the right while the spring is shortening). While one can arrive at these
geometric continuity relations on an ad hoc basis for each given system, I find it useful to follow a set procedure.

**Geometric relations:**
Let \( x_i \) denote the position of node \( i \). Let us take positive \( x \) going right. Then, we have:
(i) \( x_1 = 0 \) … since node 1 is attached to an immobile rigid wall
(ii) \( r_{sp1} = x_2 - x_1 = x_2 \) … the stretch of spring element 1 is given by the relative displacement of its two connecting nodes (note that the unstretched length of the spring can be set to zero without loss of generality… it is just a constant after all)
(iii) \( x_3 - x_2 = \text{constant} \rightarrow \dot{x}_3 = \dot{x}_2 = v_{m2} \) which follows from the fact that the mass element 2 is rigid, and so the velocity of nodes 2 and 3 must be the same as that of the mass.

**Equilibrium relations:**
Next we look at the consequences of equilibrium. This is so that we can determine relations between the forces in the various elements. To do this, we draw free-body diagrams of the various nodes (note the parallel with the finite element method in EA2). We will have to handle mass elements somewhat differently, as we shall see.

Node 1:

\[
\begin{align*}
&\text{Equilibrium says that the force exerted by the system on the wall} \\
&\text{(iv)} f_{\text{ext1}} = f_{sp1},
\end{align*}
\]

where \( f_{sp1} \) is the spring force in spring element 1.

Nodes 2-3: Since nodes 2 and 3 are attached to a rigid mass we will consider the free-body diagram of the entire block with its attached nodes. We will also put in the inertial term (mass x acceleration) as a dotted arrow.
Newton’s second law therefore says:
\[ (v) \quad -f_{sp1} = m_2 \dot{v}_{m2} \]

**Constitutive relations:**
Finally, we invoke the constitutive relations of the various elements in order to relate the various geometric quantities and the force quantities.

Spring element 1:
\[ (vi) \quad f_{sp1} = K_1 r_{sp1} \]

We now have a whole set of relations involving element-forces and element-geometric quantities. These relations contain everything we need to study the dynamics of the system, except for initial conditions which stipulate how we set the system off. All that remains for us to do now is to organize these relations in a form that is suitable for solving. This last step is called getting the state equations.

**State equations:** We chose as our state variables the spring stretch \( r_{sp1} \) and the velocity of the mass \( v_{m2} \). In fact, we will create a vector of state variables:
\[
\mathbf{X} = \begin{bmatrix} r_{sp1} \\ v_{m2} \end{bmatrix}
\]

Then we will try to get equations for the *rates* at which our state variables vary in the form:
rate of state variable = function only of state variables

Start with stretch of spring element 1:
\[
\begin{cases}
\dot{r}_{sp1} = \dot{x}_2 = v_{m2} & (\text{ii}) \\
\dot{r}_{sp1} = v_{m2} & \rightarrow \dot{r}_{sp1} = v_{m2}
\end{cases}
\]

Move on to the next state variable \( v_{m2} \):
\[
\dot{v}_{m2} = -\frac{1}{m_2} f_{sp1} = -\frac{K_1}{m_2} r_{sp1}
\]
\[
\rightarrow \dot{v}_{m2} = -\frac{K_1}{m_2} r_{sp1}
\]

Collecting all of the state equations in matrix form:

\[
\begin{bmatrix}
\dot{r}_{sp1} \\
\dot{v}_{m2}
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
-\frac{K_1}{m_2} & 0
\end{bmatrix}
\begin{bmatrix}
r_{sp1} \\
v_{m2}
\end{bmatrix}
\]

(*
which is \( \dot{X} = AX \) where \( A = \begin{bmatrix} 0 & 1 \\ -\frac{K_1}{m_2} & 0 \end{bmatrix} \) is the state matrix for the system.

In order to solve for the dynamics of our system for a given set of initial conditions, we just have to feed (*) to our Runge-Kutta solver along with the initial conditions. We have already solved this particular problem earlier, so let us now do another example.

**VI.3.2 A Spring-Damper System:** Consider a single-spring single-damper system shown below. Suppose that the spring were to be initially compressed by a certain amount and let go. What is the subsequent dynamics of the system?

![Figure 6.9: Spring-damper system](image)

**State Variables:** \( X = \{ r_{sp1} \} \) (really don’t need a matrix since we have only one)

**Geometric Relations:**

(i) \( x_1 = 0; \ x_3 = \text{constant} \)

(ii) \( r_{sp1} = x_2 - x_1 = x_2 \)

(iii) \( r_{D2} = x_3 - x_2 = -x_2 \rightarrow v_{D2} = -\dot{x}_2 \)

**Equilibrium Relations:**

Node 1: just gives me the force exerted on the wall in terms of the spring force in element 1. (and a similar equation for node 3)

\[ f_{ext1} = f_{sp1} \]

Node 2:

\[ f_{sp1} = f_{D2} \]

(iv) \( f_{D2} = f_{sp1} \)
Constitutive Relations:

\[(v) \quad f_{sp1} = K_1 r_{sp1} \]

\[(vi) \quad f_{D2} = C_2 v_{D2} \]

State Equations:

\[\dot{r}_{sp1} = \dot{x}_2 = -v_{D2} = -\frac{f_{D2}}{C_2} = -\frac{f_{sp1}}{C_2} = -\frac{K_1}{C_2} r_{sp1} \]

ie. \[\dot{r}_{sp1} = -\frac{K_1}{C_2} r_{sp1} \]

which is actually a differential equation we have seen before in Pset4! Its general solution is an exponential, in this case a decreasing exponential:

\[r_{sp1}(t) = r_o e^{-\frac{K_1}{C_2} t} \]

where \( r_o \) is the initial stretch of the spring.

Note that what this says is that if the spring is initially stretched and let go, it slowly returns to its unstretched state as the damper extends. The force in the spring (which in this case is also the force in the damper) also slowly decreases from its initial value to zero.

We can, of course, feed the above state equation to MATLAB.

```matlab
%EA3:
%
%%---------------------------------------------------------
%%Example2: Spring-damper-system
%%---------------------------------------------------------
clear all;
close all;

global A; %share this with function rate_fn which computes Xdot
%
%%%System parameters
K1 = 20; %element 1 - spring
C2 = 10; %element 2 - damper

%%%State Variables:
% X(1,:) = stretch of spring element 1

% Initial conditions
i = 1; %time index
```

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X(1,1) = 0.2;

% time parameters

start_time = 0;
dt = 0.05; % time step chosen
final_time = 5; % end calculation
time(i) = start_time; % let us keep a vector of times for plotting

% State matrix
A = [-K1/C2]; % for use in equation Xdot = A X

% Integrate equations of motion using the Runge-Kutta Method

for t = start_time+dt:dt:final_time
    k1 = rate_fn(X(:,i)); % current rate
    k2 = rate_fn(X(:,i) + dt*k1/2); % estimated mid-point rate
    k3 = rate_fn(X(:,i) + dt*k2/2); % even better mid-point rate
    k4 = rate_fn(X(:,i) + dt*k3); % excellent end-point rate

    X(:,i+1) = X(:,i) + dt*(k1/6 + k2/3 + k3/3 + k4/6); % use weighted average of rates
    time(i+1) = time(i) + dt;
    i = i+1;
end

%% Plot the stretch of spring 1
subplot(1,1,1), plot(time(:),X(1,:)), title('Stretch of Spring 1'),xlabel('Time'), grid

function rate = rate_fn(state)
% this is for use in conjunction with numerical integration schemes
% input vector contains "state" X
% output vector "rate" sends back Xdot

global A; % share A matrix with main program
rate = A*state;

Figure 6.10: Exponential response of a spring-damper system