VI.5 FORCED OSCILLATIONS OF A SPRING-MASS SYSTEM

Let us return to the spring-mass system. But now let us suppose that a *force source* acts at one end of the mass as shown. Furthermore let us say that the spring-mass system is initially quiescent (spring is unstretched and mass has zero velocity at time zero).

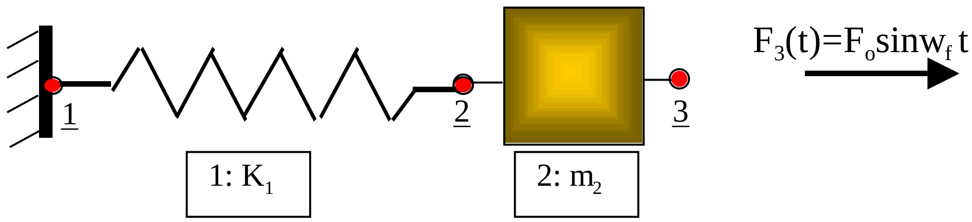


Figure 6.15: Spring-mass system with external forcing

State variables: $\underline{X} = \begin{Bmatrix} r_{sp1} \\ v_{m2} \end{Bmatrix}$

It should be easy for you to show that the state equation for this system is now given by:

$$\begin{Bmatrix} \dot{r}_{sp1} \\ \dot{v}_{m2} \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{K_1}{m_2} & 0 \end{bmatrix} \begin{Bmatrix} r_{sp1} \\ v_{m2} \end{Bmatrix} + \begin{Bmatrix} 0 \\ \frac{F_3(t)}{m_2} \end{Bmatrix}$$

which I will write as:

$$\underline{\dot{X}} = \underline{A}\underline{X} + \underline{F}$$

where $\underline{F}$ is called the source term.

In prinicple, it should not be any harder to solve the above system of equations using our MATLAB m-files, but I will need to modify my rate function file (rate_fn.m) to another one (frate_fn.m) which can handle the additional source term on the right side. I have done this for you.

```
%EA3
%%-------------------------------------------------------
%%Example: Spring-mass-system with a harmonic force source
%%-------------------------------------------------------
clear all;
close all;

global A F;                    %share these with function frate_fn which computes Xdot=AX+F
%%
%%System parameters
K1      =       40;                             %element 1 - spring
m2      =       10;                             %element 2 - mass
wf      =       2.0001;                 %frequency of force source function F3(t)=F0*sin(wf*t)
F0      =       1;                              %force source amplitude
```

```
%State Variables:
%	X(1,:) = stretch of spring element 1
%	X(2,:) = velocity of mass element 2

% Initial conditions quiescent.  Motion only due to force source
i	=	1;					%time index
X(1,1) = 0;
X(2,1) = 0;
%time parameters

ws=sqrt(K1/m2);				%natural frequency of oscillation of unforced system
period=2*pi/ws;				%period of oscillation

%analytical solution
an_fac = (F0/m2)*(1/(ws^2-wf^2));		%analytical solution for spring stretch
an_X(1) = 0;

start_time = 0;
dt	=	0.01*period;			%time step chosen as a fraction of one period
final_time	=	40*period;		%end calculation after ten periods
time(i)	=	start_time;			%let us keep a vector of times for plotting

%State matrix
A = [0,		1;
	-K1/m2,0];  % for use in equation Xdot = AX+F
%
% Integrate equations of motion using the Runge-Kutta Method

for t = start_time:dt:final_time

    F = [0,	F0*sin(wf*t)/m2]';			% current force source value
    k1 = frate_fn(X(:,i));			% current rate
    k2 = frate_fn(X(:,i) + dt*k1/2);		% estimated mid-point rate
    k3 = frate_fn(X(:,i) + dt*k2/2);		% even better mid-point rate
    k4 = frate_fn(X(:,i) + dt*k3);  % excellent end-point rate

    X(:,i+1) = X(:,i) + dt*(k1/6 + k2/3 + k3/3 + k4/6);	% use weighted average of rates
    time(i+1) = time(i) + dt;
	%analytical solution
	an_X(i+1) = an_fac*(sin(wf*time(i+1))-(wf/ws)*sin(ws*time(i+1)));
	i = i+1;
end

%%Plot the stretch of spring 1
subplot(2,1,1), plot(time(:),X(1,:)), title('Stretch of Spring 1'), grid
subplot(2,1,2), plot(time(:),an_X(:)), title('Analytical'),xlabel('Time'), grid
```

```
function rate = frate_fn(state)
% this is for use in conjunction with numerical integration schemes
% input vector contains"state" X
% output vector "rate" sends back Xdot

global A F;			% share A matrix with main program
rate = A*state+F;
```

Analytical Solution: It is easy to get an analytical solution to this problem as well. Once again, in trying to get analytical solutions, we will get a second-order DE for the spring stetch r_{sp1}:

$$\frac{d^2 r_{sp1}}{dt^2} + \frac{K_1}{m_2} r_{sp1} = F_o \sin\omega_f t \qquad (f)$$

The only difference between this equation and the earlier spring-block oscillator equation is that there is something on the right side. In fact, if the right side had been zero, we know that the solution should go as:

$$r_{sp1}(t)_{no\_source} = A\sin\omega_n t + B\cos\omega_n t \qquad \text{(ß)}$$

where $\omega_n = \sqrt{\frac{K_1}{m_2}}$ is the angular frequency of the system without the force source (also called the *natural* frequency).

If we try the above solution (ß) in the DE (f), we will find that the left side will cancel out, leaving something on the right side! Therefore we need to *add* something to (ß) in order to get the correct solution. I am going to guess that this something is going to look like:

$$r_{sp1}(t)\Big|_{source} = D\sin\omega_f t$$

where D is some constant. To see if this is correct, we substitute the above in the DE (f):

$$-\omega_f^{\,2} D\sin\omega_f t + \omega_n^{\,2} D\sin\omega_f t = \frac{F_o}{m_2}\sin\omega_f t$$

which says that D must be:

$$D = \left\{\frac{F_o/m_2}{\omega_n^2 - \omega_f^2}\right\}\sin\omega_f t \qquad \text{(ßß)}$$

In fact, the general solution to (f) is actually the sum of (ß) and (ßß). Go ahead and try it.

$$r_{sp1}(t) = A\sin\omega_n t + B\cos\omega_n t + \left\{\frac{F_o/m_2}{\omega_n^2 - \omega_f^2}\right\}\sin\omega_f t$$

For quiescent initial conditions, $r_{sp1}(0)=0$ and $v_{sp1}(0)=0$, we find that the stretch of the spring is given by:

$$r_{sp1}(t) = \left\{\frac{F_o/m_2}{\omega_n^2 - \omega_f^2}\right\}\left\{\sin\omega_f t - \frac{\omega_f}{\omega_n}\sin\omega_n t\right\}$$

Remarks:

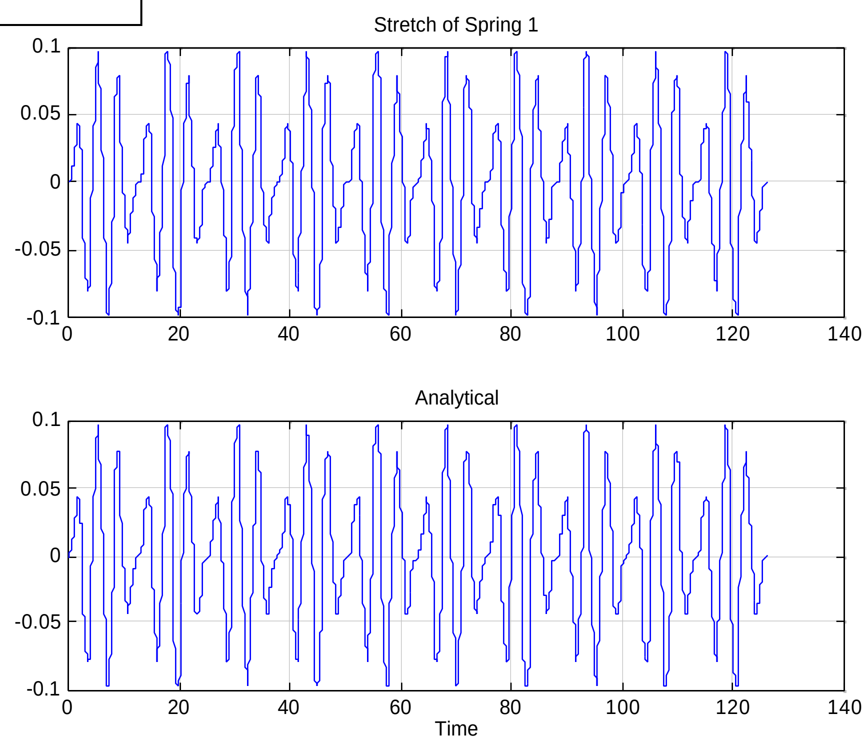
1. The system starts moving even though we have quiescent initial conditions because of the external forcing. Earlier, we set the system off from its equilibrium state, and then

watched how the system responded. Here, we start with the system in its quiescent state and see how it responds to time varying external stimuli.

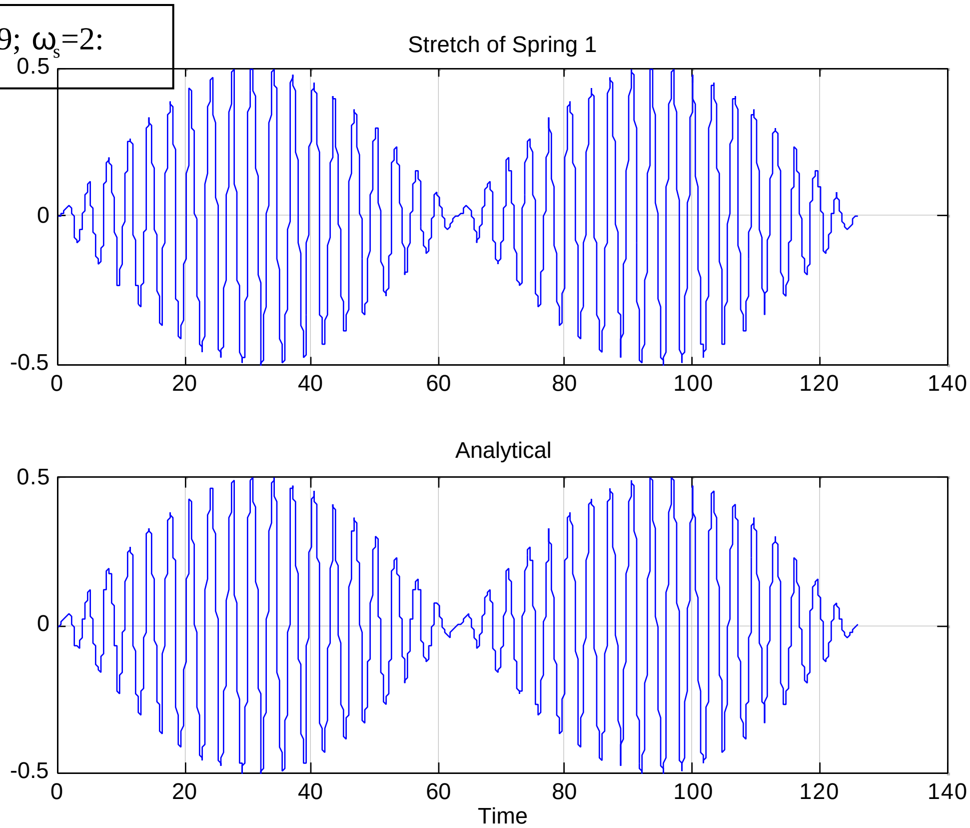
2. Note that as the forcing frequency approaches the natural frequency of the spring-mass system, the amplitude of oscillations dramatically increases (actually becomes infinite)! This phenomenon is called *resonance*. One could have catastrophic motions build up if the external stimuli to the system approaches the natural frequencies of the system

I have coded the analytical solution as well in the MATLAB file for you to see how it compares with the numerical solutions. The system behavior for different forcing frequencies are shown below.

ω_f=1.5; ω_s=2:



ω_f=1.9; ω_s=2:



ω_f=1.95; ω_s=2: