IV Work and Energy

Newton's laws provide us the tools necessary to track the motion of a body under the action of external forces. We have seen how to go about that for several situations, some of which were amenable to analytical solutions, and other cases could only be solved numerically. We will now explore some additional concepts of classical mechanics that provide us more insight into the workings of nature.

IV.1 Work and Kinetic Energy:

Let **F** be the net force exerted on a body by something external to it. We define the **work** U *done by the force* **F** in moving the body through some distance from point $\mathbf{r}_i$ to point $\mathbf{r}_f$ as:

$$U = \int_{\mathbf{r}_i}^{\mathbf{r}_f} \mathbf{F}.\mathbf{dr} \qquad (4.1)$$

where the integral is taken along the path of the motion. The units of work are evidently N.m (which is also known as the **joule** J) in SI units. Note that by definition (because of the dot product) it is only the component of the force that acts tangential to the path that contributes to the work.

Next we define the **kinetic energy** of a body in motion as:

$$K = \frac{1}{2} m|\mathbf{v}|^2 \qquad (4.2)$$

where m is the mass of the body and **v** is the instantaneous velocity. You can check that the dimensions of energy are the same as those of kinetic energy, and therefore the unit of kinetic energy is also the joule.

Because of the force **F** acting on it, the body accelerates and so its velocity changes from say $\mathbf{v}_i$ to $\mathbf{v}_f$.. According to Newton's second law, we have:

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt} \tag{4.3}$$

Let us now do some algebraic manipulation of the above. First, take the dot product of the above equation with **v**:

$$\mathbf{F}.\mathbf{v} = m\frac{d\mathbf{v}}{dt}.\mathbf{v} \tag{4.4}$$

Note that we can use the product rule of differentiation to get a useful identity:

$$\frac{d}{dt}(\mathbf{v}.\mathbf{v}) = \frac{d\mathbf{v}}{dt}.\mathbf{v} + \mathbf{v}.\frac{d\mathbf{v}}{dt} = 2\frac{d\mathbf{v}}{dt}.\mathbf{v} \tag{4.5}$$

and so (4.4) can be re-written as:

$$\mathbf{F}.\frac{d\mathbf{r}}{dt} = \frac{1}{2}m\frac{d}{dt}\left(|\mathbf{v}|^2\right) \tag{4.6}$$

where we have recast the velocity on the left side in terms of the rate of change of position, and we have used the identity (4.5) to recast the right side in terms of the magnitude of the velocity.

Now, since (4.6) holds true over any time interval dt we must have:

$$\mathbf{F}.d\mathbf{r} = \frac{1}{2}m\ d\left(|\mathbf{v}|^2\right) \tag{4.7}$$

which upon integration from the initial point to the final point gives:

$$\begin{aligned}\int_{\mathbf{r}_i}^{\mathbf{r}_f}\mathbf{F}.d\mathbf{r} &= \int_{\mathbf{v}_i}^{\mathbf{v}_f}\frac{1}{2}m\ d\left(|\mathbf{v}|^2\right) \\ &= \frac{1}{2}m\left(|\mathbf{v}_f|^2 - |\mathbf{v}_i|^2\right)\end{aligned} \tag{4.8}$$

That is:

$$U = \Delta K \tag{4.9}$$

which reads: the work done by the net external force acting on a body is equal to the resulting change in the kinetic energy of the body.

Remarks:

(i) This is called the work-energy theorem. Note that this is a scalar equation unlike Newton's laws which were vector equations. Therein lies the utility of looking at dynamics from an energy point of view. The work-energy theorem provides us a simpler way of extracting certain key features of the motion of a body without having to work out the motion in detail.

(ii) The work-energy theorem is not a new law, but a direct consequence of Newton's laws. However, energy considerations do lead us to a new truth called the conservation of energy and this is a great truth because, with appropriate interpretation, it appears to hold universally for all kinds of systems (not only for classical mechanical systems). We will get to that shortly.

Example 1: **Bullet shot straight up in near-earth gravity (no air drag)**

Let a bullet of mass m be fired straight up from a rifle. If the bullet is fired with an exit speed of V_0 what is the height H it will reach before falling back to earth? Neglect air drag. You know how to solve this starting with Newton's laws, but let us use the work-energy theorem here. Consider our usual coordinate frame (x-axis along flat earth, and y-axis vertically up; gravity acts along negative y).

The net force acting on the ball is: $\mathbf{F} = -mg\mathbf{j}$ which stays constant over the entire motion.

The work done on the ball by the force of gravity is:

$$U = \int_{\mathbf{r}_i}^{\mathbf{r}_f} \mathbf{F}.d\mathbf{r} = \int_0^H (-mg).dy = -mgH$$

The change in kinetic energy of the ball is:

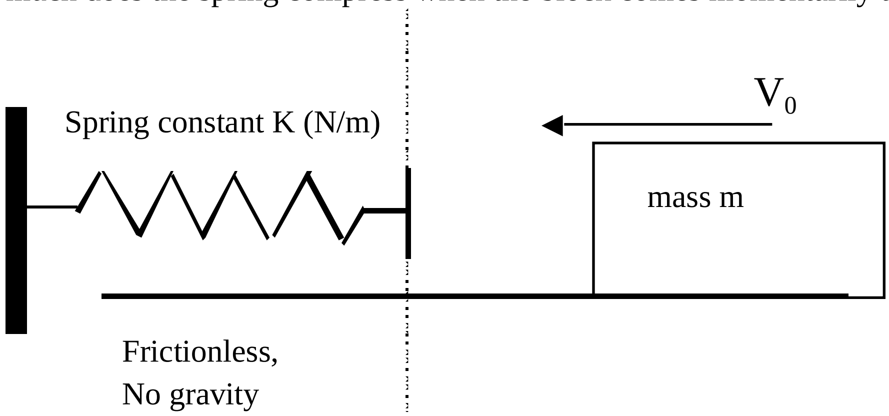
$$\Delta K = \frac{1}{2}m\left(|\mathbf{v}_f|^2 - |\mathbf{v}_i|^2\right) = -\frac{1}{2}m|V_0|^2$$

From the work-energy theorem, we therefore have:

$$H = \frac{V_0^2}{2g}$$

Example 2: Block running into a linear spring

A block of mass m is moving left with a speed V_o on a frictionless horizontal surface. It is to be brought momentarily to rest by the linear spring of spring constant K. By how much does the spring compress when the block comes momentarily to rest?



The change in kinetic energy is: $\Delta K = \frac{1}{2} m V_0^2$

The work done by the force exerted by the spring is:

$$U = \int_{\mathbf{r}_i}^{\mathbf{r}_f} \mathbf{F}.\mathbf{dr} = \int_0^{r_{xf}} (-K r_x) dr_x = -\frac{1}{2} K r_{xf}^2$$

The negative sign is because the force exerted by the spring acts opposite the direction in which the block moves. And so from the work-energy theorem we get:

$$r_{xf} = V_0 \sqrt{\frac{m}{K}}$$

This is the amount that the spring compresses when the block comes momentarily to rest.
Remark: How does this compare with the analytical solution that we got earlier?

IV.2 Potential Energy:

In both the examples above, the initial kinetic energy of the moving body (block or the bullet) goes to zero. In the first example, the bullet then comes falling back to earth, and in the second example the spring pushes the block back to the right. It is useful to think of this in the following way. Imagine that the kinetic energy has somehow been transferred to a different kind of energy that is stored in the earth-ball system or in the spring-block system respectively. This energy we call the **potential** energy. Potential energy can be recovered back into kinetic energy as happens when the bullet falls back towards earth or when the block gets pushed back to the right. The potential energy of a system depends only on the configuration of a system and does not depend on how it got there.

It is not always true that the kinetic energy of a system gets transferred into a potential energy. Sometimes the kinetic energy is lost due to friction as heat (which is motion of atoms and molecules). It turns out that for a certain class of forces called **conservative** forces, the work done by these forces in going from one state to another is path independent, and depend only on the end states. For such conservative forces, it is always possible to associate a potential energy.

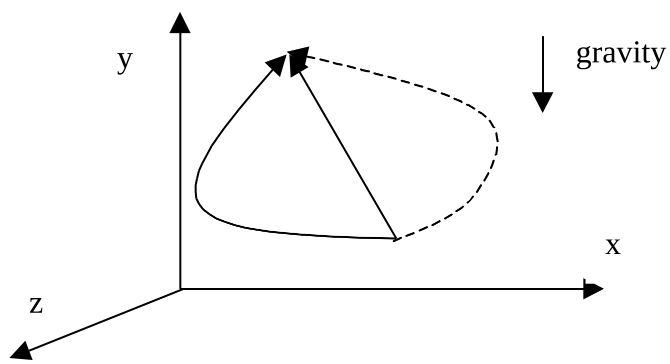


Figure: Work due to gravity

Consider the force due to earth's gravity for instance. Suppose a body were to move from $\mathbf{r}_i$ to $\mathbf{r}_f$ and the force acting on it is due to earth's gravity: **F**=-mg**j**. It is easy to see that the work done by this force is:

$$U = -mg\left\{ r_{yf} - r_{yi} \right\} \tag{4.10}$$

irrespective of the path of the particle. Therefore the gravitational force is a conservative force by our definition.

For conservative forces, it is always possible to define a function $\Phi(\mathbf{r})$ which depends only on position, such that the work done on an object by these forces in moving through **dr** is :

$$\mathbf{F.dr} = -d\Phi \tag{4.11}$$

Therefore, the work done in moving from state $\mathbf{r}_i$ to state $\mathbf{r}_f$ is given by:

$$U = \int_{\mathbf{r}_i}^{\mathbf{r}_f} \mathbf{F.dr} = \int_{\Phi_i}^{\Phi_f} -d\Phi = \Phi_i - \Phi_f \tag{4.12}$$

For the near-earth gravitational force we can associate a potential energy function function $\Phi = \Phi_o + mgr_y$ which depends only on position where Φ_o is any arbitrary constant (usually taken as zero which simply says that the gravitational potential energy at some reference position or "datum" $r_y=0$ is zero).

The force due to an elastic spring turns out to be conservative as well. To find the associated potential energy function, we seek a Φ such that: $\mathbf{F.dr} = -d\Phi$

For a linear spring this yields:

$$\begin{aligned} (-Kr_x)\ \ dr_x &= -d\Phi \\ \int (-Kr_x)\ \ dr_x &= \int -d\Phi \\ \Phi &= \Phi_0 + \frac{1}{2}Kr_x^2 \end{aligned} \tag{4.13}$$

It is usual to take the potential energy when the spring is relaxed to be zero, and it is usual to denote the amount of stretch or compression of the spring by r_{sp}. So the potential energy function associated with the deformation of a linear spring is given by:

$$V(r_{sp}) = \frac{1}{2}Kr_{sp}^2 \tag{4.14}$$

IV.3 Conservation of Energy:

For systems with only conservative forces acting on them, we find that:

Work done: $U = \int_{\mathbf{r}_i}^{\mathbf{r}_f} \mathbf{F.dr} = \int_{\Phi_i}^{\Phi_f} -d\Phi = \Phi_i - \Phi_f$

(4.15)

Change in kinetic energy: $\Delta K = \frac{1}{2} m\left(|\mathbf{v}_f|^2 - |\mathbf{v}_i|^2\right)$ (4.16)

and so from the work-energy theorem:

$$\Phi_i - \Phi_f = \frac{1}{2} m\left(|\mathbf{v}_f|^2 - |\mathbf{v}_i|^2\right) \tag{4.17}$$

which is better written as:

$$\Phi_i + \frac{1}{2} m|\mathbf{v}_i|^2 = \Phi_f + \frac{1}{2} m|\mathbf{v}_f|^2 \tag{4.18}$$

This is equivalent to saying that:

Potential energy + Kinetic energy = constant (4.19)

since the initial and final states are just any two states, and so the above holds for any state.

Remarks:

(i) This is a statement of conservation of energy that holds for systems which have only conservative forces acting on them.

(ii) Actually the work done by non-conservative forces (friction, which we will see shortly, is one such creature) etc go into other kinds of energy (such as heat), and if we account for all such energies we come up with a grand law of nature:

Total energy of a closed system = constant

Energy considerations for the spring-block oscillator system: For the undamped spring-block oscillator, the conservation of energy can be written as:

$$\frac{1}{2}Kr_{sp}^2 + \frac{1}{2}mv^2 = \mathrm{E} \quad (4.20)$$

where v is the speed of the block, and E is called the mechanical energy of the spring-block system. When the spring is unstretched, the energy is all in the block as kinetic energy, and the kinetic energy decreases as the block is slowed down by the spring. At the same time, since the spring gets compressed or stretched, the potential energy in the spring increases correspondingly till the time when the block comes to rest when the mechanical energy of the system is all stored in the spring as potential energy. On the return trip of the block, the potential energy in the spring decreases while increasing the kinetic energy of the block…etc.

Energy considerations for the earth-bullet system: The mechanical energy of the earth-bullet system of example 1 can be written as:

$$mgr_y + \frac{1}{2}mv^2 = \mathrm{E} \quad (4.21)$$

where r_y is the height of the bullet above a chosen reference, which in this case is the surface of the earth. As the bullet moves up, the kinetic energy of the bullet is transferred as potential energy in the earth-bullet system till the bullet comes to a halt at height H when it is all potential energy. As the bullet comes down, the reverse happens. Unlike the spring-block oscillator, however, this system does not keep going back and forth.

IV. 3 Power:

We define the **power** as the rate at which work is done. So if a force **F** acting on a body were to move it by a distance **dr** in time dt, we have the power:

$$P = \mathbf{F}.\frac{d\mathbf{r}}{dt} = \mathbf{F}.\mathbf{v} \quad (4.22)$$

It is easy to see form (4.6) that:

$$P = \frac{d}{dt}\left\{ \frac{1}{2} m|\mathbf{v}|^2 \right\} \quad (4.23)$$

which states that the rate of change of kinetic energy is equal to the power transferred to the body.