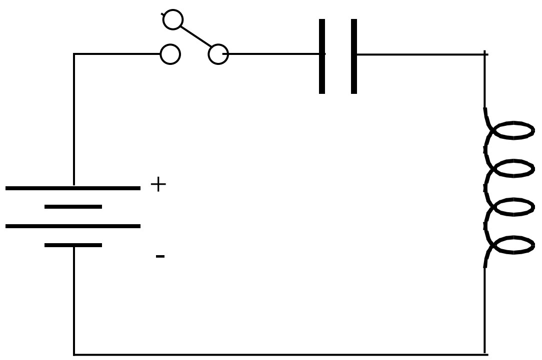
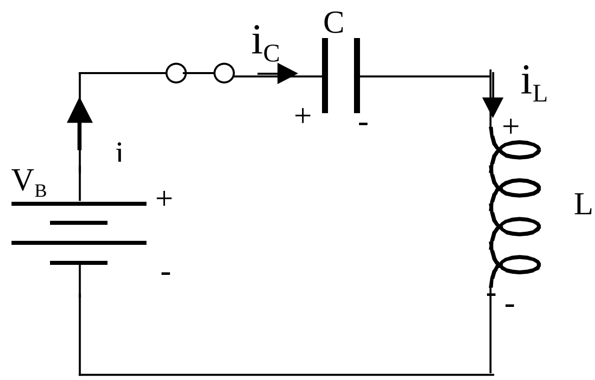
VIII.2.1 LC-Oscillator Circuit: At time t=0, the switch is closed. A current begins to flow and the capacitor charges. How do the voltages and currents in the circuit change with time?





State Variables: $\underline{X} = \{V_C \quad i_L\}^T$

KCL: (i) $i = i_C = i_L$

KVL: (ii) $V_C + V_L = V_B$ (loop:C-L-battery)

Constitutive Equations: (iii) $V_C = \frac{1}{C} q_C$

(iv) $V_L = L\frac{di_L}{dt}$

State Equations:

$$\frac{dV_C}{dt} \overset{(iii)}{=} \frac{1}{C}\frac{dq_C}{dt} \overset{defn}{=} \frac{1}{C} i_C \overset{(i)}{=} \frac{1}{C} i_L$$

$$\frac{di_L}{dt} \overset{(iv)}{=} \frac{1}{L} V_L \overset{(ii)}{=} \frac{1}{L}\{V_B - V_C\}$$

which in matrix form becomes: $\begin{Bmatrix} \frac{dV_C}{dt} \\ \frac{di_L}{dt} \end{Bmatrix} = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & 0 \end{bmatrix} \begin{Bmatrix} V_C \\ i_L \end{Bmatrix} + \begin{Bmatrix} 0 \\ V_B/L \end{Bmatrix}$

One can solve this numerically given the initial conditions.

We also know how to solve this analytically. Let us get the second-order state equation for this system using the voltage across the capacitor as the dynamic variable.

$$\frac{d^2V_C}{dt^2} + \frac{1}{LC}V_C = \frac{1}{LC}V_B$$

By now you should be able to figure out the solution to the above differential equation is an oscillatory function with a correction term for the source term on the right side.

$$V_C(t) = V_B + A\sin\omega_n t + B\cos\omega_n t$$

where $\omega_n = 1/\sqrt{LC}$ is the natural angular frequency of this LC-oscillator.

If at time t=0, the switch is turned on, the voltage must be zero, and since the inductor is an “inertial” it cannot change its current instantaneously, we must have the current in the inductor is zero as well at this initial instant. Note that since in this circuit: $i_L = i_C = \frac{dq_C}{dt} = \frac{1}{C}\frac{dV_C}{dt}$ what this means is that not only is the voltage across the capacitor equal to zero at time zero, but also the first derivative of the voltage across the capacitor. This provides us the necessary conditions to obtain the constants A and B.

$$V_C(t) = V_B\{1 - \cos\omega_n t\}$$

The voltage across the inductor is:

$$V_L(t) = V_B\cos\omega_n t$$

The current in the capacitor and inductor is given by:

$$i(t) = \frac{V_B}{C}\omega_n\sin\omega_n t$$