COMPLEX EXPONENTIALS

Consider the function: \( f(t) \)
Let it be a well-behaved function (for which derivatives of all orders necessary exist everywhere).

Functions like \( f(t) = \sin \omega t \); \( f(t) = e^{\omega t} \) are acceptable.
Functions like \( |\omega t| \) are not.

Note that in Calculus you learnt (or should have learned) that such functions can be expanded in a Taylor series as follows:

\[
f(t) = f(0) + \frac{df}{dt}(0) t + \frac{d^2f}{dt^2}(0) \frac{t^2}{2!} + \frac{d^3f}{dt^3}(0) \frac{t^3}{3!} + \ldots + \frac{d^n f}{dt^n}(0) \frac{t^n}{n!} + \ldots
\]

where \( n! = n(n-1)(n-2)\ldots(3)(2)(1) \) is read as (n-factorial);
and the derivatives are all evaluated at \( t=0 \). That is, most well-behaved functions can be expressed in the form of a power series!

For example, it should be straightforward for you to obtain the following Taylor series expansions for some common functions:

\[
e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \ldots
\]

\[
\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots
\]

\[
\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots
\]

(I am using \( x \) instead of \( t \) now and soon will use something else, but these are just symbols representing the argument of the function under consideration).

Now the meaning of a complex exponential function becomes clear. I just need to set \( x=j\theta \) in the above to get:

\[
e^{j\theta} = 1 + (j\theta) + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \frac{(j\theta)^4}{4!} + \ldots + \frac{(j\theta)^n}{n!} + \ldots
\]

\[
= \left\{ \left[1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots \right] + j\left[ \frac{\theta^3}{3!} - \frac{\theta^5}{5!} + \frac{\theta^7}{7!} + \ldots \right] \right\}
\]

\[
= \cos \theta + j\sin \theta
\]

where I have used the fact that \( j = \sqrt{-1} \); \( j^2 = -1 \); \( j^3 = j^2 j = -j \); \( j^4 = +1 \); ...

That is, the complex exponential is related to the trigonometric functions: sine and cosine.
We have just derived Euler’s formula.