THEORY OF ELASTICITY

4.1 THE FIELD EQUATIONS OF LINEAR ELASTICITY

We are now in a position to put our theory together. The list of unknowns (pay particular attention to how I stack them) that we need to solve for in our theory consists of:

displacements (3): $\mathbf{u}^{\mathbf{T}} = \left[u_x \quad u_y \quad u_z \right]$

strains (6): $\varepsilon^{\mathbf{T}} = \left[\varepsilon_{xx} \quad \varepsilon_{yy} \quad \varepsilon_{zz} \quad \gamma_{xy} \quad \gamma_{yz} \quad \gamma_{zx} \right]$

where it is important to bear in mind that we are now using *engineering* shear strains which are related to their tensorial counterparts through: $\gamma_{xy} = 2\varepsilon_{xy},\ \gamma_{yz} = 2\varepsilon_{yz},\ \gamma_{zx} = 2\varepsilon_{zx}$.

and stresses (6): $\sigma^{\mathbf{T}} = \left[\sigma_{xx} \quad \sigma_{yy} \quad \sigma_{zz} \quad \sigma_{xy} \quad \sigma_{yz} \quad \sigma_{zx} \right]$

Note also that I am now stacking the stresses and strains as column vectors - this is useful when we do numerical solutions.

Let us now collect the complete set of equations of linear elasticity.

EQUILIBRIUM EQUATIONS (3):

$$\begin{aligned}
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + f_x &= \rho \ddot{u}_x \\
\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + f_y &= \rho \ddot{u}_y \\
\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + f_z &= \rho \ddot{u}_z
\end{aligned} \qquad (1)$$

where a superposed dot represents a time derivative, and ρ is material density.

LINEAR ELASTIC, ISOTROPIC MATERIAL RESPONSE (6): $\sigma = \mathbf{D}\varepsilon$ where

$$\mathbf{D} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 0 & 0 & 0 \\ \frac{\nu}{1-\nu} & 1 & \frac{\nu}{1-\nu} & 0 & 0 & 0 \\ \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)} \end{bmatrix} \qquad (2)$$

STRAIN-DISPLACEMENT RELATIONS (6):

$$\begin{aligned}
\varepsilon_{xx} &= \frac{\partial u_x}{\partial x} & \gamma_{xy} &= \gamma_{yx} = \left\{ \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right\} \\
\varepsilon_{yy} &= \frac{\partial u_y}{\partial y} & \gamma_{yz} &= \gamma_{zy} = \left\{ \frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z} \right\} \\
\varepsilon_{zz} &= \frac{\partial u_z}{\partial z} & \gamma_{xz} &= \gamma_{zx} = \left\{ \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \right\}
\end{aligned} \qquad (3)$$

In lieu of the strain-displacement relations, we may use the compatibility conditions for the strains. For any given problem, we must solve for the fifteen unknowns using the above set of fifteen equations subject to the appropriate boundary constraints (either applied forces or displacements).

As you can imagine, it is not a trivial matter to solve these equations in general. So we make further approximations to reduce the complexity of the problem. These are typically based on our observations about the geometry of the problem, and the resulting simplified theories will be applicable in a much more restricted and often approximate sense only. In this course we will make simplifications that lead to the theories of: (i) plane stress, (ii) plane strain, (iii) Bernoulli-Euler beam theory, and (iv) Kirchoff's Plate Theory.

4.2 TWO DIMENSIONAL PROBLEMS IN CARTESIAN COORDINATES: PLANE STRESS AND PLANE STRAIN

Plane Stress: For certain thin plate like structures where the loading is in the plane of the plate (say in the xy plane) and nothing varies through the thickness of the structure, the only significant non-zero stress components are:

$$\sigma^{T} = \begin{bmatrix} \sigma_{xx} & \sigma_{yy} & \sigma_{xy} \end{bmatrix}$$

and furthermore these are essentially uniform through the thickness z-direction. Such a state of stress is called plane stress. Plane stress is a useful two-dimensionalization that is appropriate in some cases as we will see later on.

- In the case of plane stress, the equations of static equilibrium for the case of no body forces, immediately simplify to:

$$\begin{aligned} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} &= 0 \\ \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} &= 0 \end{aligned} \qquad (4)$$

- From the material response relations we find:

$$\varepsilon_{zz} = -\frac{\nu}{E}\left(\sigma_{xx} + \sigma_{yy}\right) \qquad (5)$$

The above tells us that if we want a structure to be in a state of plane stress, then we must allow out-of-plane normal straining which is given by the above "auxiliary relation" in terms of the in-plane normal stresses.

The rest of the stress-strain relations then reduce to:

$$\sigma = \mathbf{D}\varepsilon \quad \text{where} \quad \mathbf{D} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \qquad (6)$$

where $\varepsilon^{T} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{yy} & \gamma_{xy} \end{bmatrix}$.

- The in-plane strain-displacement relations are:

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}, \quad \varepsilon_{yy} = \frac{\partial u_y}{\partial y}, \quad \gamma_{xy} = \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) \tag{7}$$

- The list of unknowns for the case of plane stress reduces to:

$$\{\sigma_{xx}, \sigma_{yy}, \sigma_{xy}, \varepsilon_{xx}, \varepsilon_{yy}, \gamma_{xy}, u_x, u_y; \ \varepsilon_{zz}, u_z\}$$

Given a problem with boundary conditions, the first eight of these ("in-plane") unknowns: $\sigma^T = [\sigma_{xx} \quad \sigma_{yy} \quad \sigma_{xy}]$, $\varepsilon^T = [\varepsilon_{xx} \quad \varepsilon_{yy} \quad \gamma_{xy}]$, and $\mathbf{u}^T = [u_x \quad u_y]$ can be determined using the two equilibrium equations (4), the three reduced stress-strain relations (6), and the three in-plane strain-displacement relations (7). The problem is therefore entirely two-dimensional in this sense. Having determined the in-plane quantities, we can then use the `auxiliary' relation (5) to determine the out-of-plane normal strain ε_{zz} and the corresponding out-of-plane displacement u_z.

- The above set of equations can be further compacted through some algebraic manipulations.

(i) It is convenient to define an **Airy stress function** $\Phi(x,y)$ as follows:

$$\sigma_{xx} = \frac{\partial^2 \Phi}{\partial y^2}, \quad \sigma_{yy} = \frac{\partial^2 \Phi}{\partial x^2}, \quad \sigma_{xy} = -\frac{\partial^2 \Phi}{\partial x \partial y} \tag{8}$$

Then the equilibrium equations (4) are automatically satisfied by $\Phi(x,y)$ (see Pset 1).

(ii) Eliminate the displacements from the strain-displacement relations to get the compatibility relation (see chapter 3):

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 2 \frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} \tag{9}$$

(iii) Use the reduced stress-strain relations (6) to express the strains in terms of the stresses in (9). Then replace the stresses in favor of the Airy stress function $\Phi(x,y)$ using (8) to get:

$$\boxed{\nabla^4 \Phi \equiv \frac{\partial^4 \Phi}{\partial x^4} + \frac{\partial^4 \Phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \Phi}{\partial y^4} = 0}. \tag{10}$$

The above is called the **bi-harmonic equation**. This approach considerably simplifies our bookkeeping. All we have to do now is find $\Phi(x,y)$ for a given problem using the above equation and the appropriate boundary conditions! Having done so, it is a simple matter to obtain the stresses (which will then automatically satisfy equilibrium), from which we can get the strains (which will automatically satisfy the compatibility relations). Before we get too excited, however, we should note that to obtain $\Phi(x,y)$ we have to solve a fourth-order partial differential equation (pde)! That is not always an easy task. Of course, some simple solutions to (10) might be obvious to you (such as certain third-order polynomials). You can also *learn* to solve the bi-harmonic equation (10) for more complicated solutions. Or else, you can look up the Airy stress functions in handbooks on stress analysis for many problems of practical interest. If all else fails, you can get yourself promoted to the managerial ranks and hire an underemployed applied mathematician to solve the pde for you!

<u>Plane Strain</u>: For certain thick bodies subject to only in-plane loading and where nothing varies along the thickness (z-direction) we may assume that plane deformation holds. That is: $u_z = 0$ and the only non-zero displacements are $\mathbf{u}^T = \begin{bmatrix} u_x & u_y \end{bmatrix}$. We can reduce the complete set of field equations of elasticity under the plane deformation assumption just as we did for plane stress. I am just going to give you a brief outline here for the case of static equilibrium under no body forces. You can fill in the details if you wish.

- From the strain-displacement relation, we are immediately led to the conclusion that the only non-zero strains are: $\varepsilon^T = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{yy} & \gamma_{xy} \end{bmatrix}$, and all the other components are zero. Therefore plane deformation is also known as **plane-strain**.

- The stress-strain relations can also be reduced for plane strain. In particular, note that $\varepsilon_{zz} = 0$ implies that we must have out-of-plane normal stress component:

$$\sigma_{zz} = \nu\left(\sigma_{xx} + \sigma_{yy}\right) \qquad (11)$$

which is given in terms of the in-plane stresses by the above auxiliary relation for plane strain. In fact, apart from the above stress component, the only other non-zero stress components are: $\sigma^T = \begin{bmatrix} \sigma_{xx} & \sigma_{yy} & \sigma_{xy} \end{bmatrix}$, and these are related to the in-plane strains through:

$$\sigma = \mathbf{D}\varepsilon \text{ where } \mathbf{D} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1 & \nu/(1-\nu) & 0 \\ \nu/(1-\nu) & 1 & 0 \\ 0 & 0 & (1-2\nu)/2(1-\nu) \end{bmatrix} \quad (12)$$

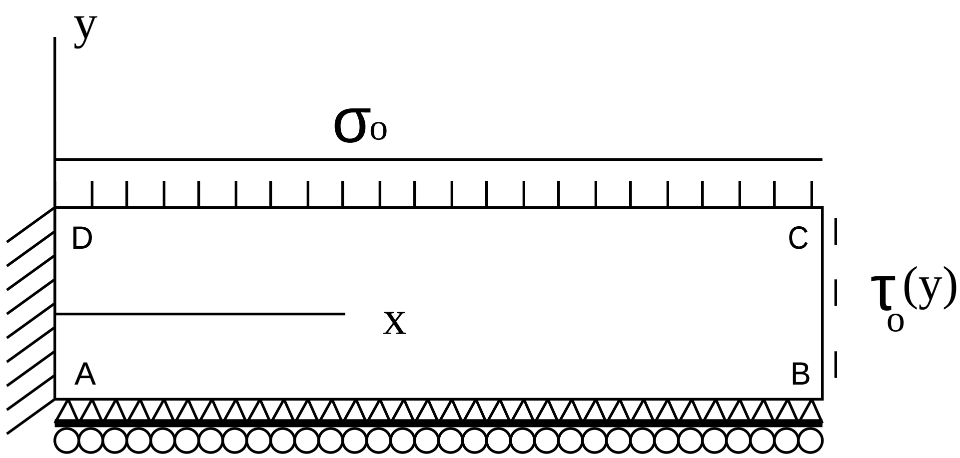
- You should be able to see that the equilibrium relations are the same as for plane-stress (4). Invoking an Airy stress function as before, we find -- after some straightforward and maybe tedious algebra -- that plane strain equations are very identical to that of plane stress except for the auxiliary relation and the reduced stress-strain relations. Keep in mind thought that they are entirely different kinds of "two-dimensional" approximations valid for certain extreme situations of very thin or very thick bodies.

4.3 PLANE PROBLEMS IN POLAR COORDINATES: As we have noted before, sometimes it is best to use a cylindrical polar coordinate system. The equations of plane stress and plane strain in cylindrical polar coordinate system are collated in the handout I gave you in class. You might also want to look through the appropriate problems in Homework sets 1 and 2 dealing with polar coordinates.

4.4 SPECIFICATION OF BOUNDARY CONDITIONS:

We have in our bag of tricks all the equations of isotropic linear elasticity needed for general three-dimensional stress analysis, as well as the reduced equations applicable to two-dimensional plane stress and plane strain problems. All we now need to do is figure out how to specify the boundary conditions appropriate to any given structural geometry and loading. Note that only the boundary conditions and the actual values for the material properties will change from problem to problem. We will see how to prescribe boundary conditions by means of several examples. Let us restrict attention to two-dimensions for simplicity.

Example 1: Consider the two-dimensional problem shown in the figure.



Boundary AD: This boundary is embedded in a rigid wall and so is not allowed to displace. That is

$$u_x|_{AD} = 0 \; , \; u_y|_{AD} = 0.$$

We cannot say anything about the stresses on this boundary. They will be whatever the support can provide as reaction to keep the entire structure in equilibrium.

Boundary AB: Here,

$$u_y|_{AB} = 0, \; \sigma_{yx}|_{AB} = 0$$

This is because this boundary is constrained by rollers in such a manner that it is allowed to displace in the x-direction but not in the y-direction. Since the y-direction displacement is constrained, we cannot specify the corresponding (y-directed) normal stress σ_{yy} on this boundary. Also, since the x-displacement is not constrained, we can specify (or apply) any (x-directed) shear stress on this boundary. For the given example, no shear stress is applied, and so we have the second condition above.

Boundary BC: The displacement of this boundary is not constrained in any way. There is no normal stress applied, and there is a shear stress acting on it. Therefore:

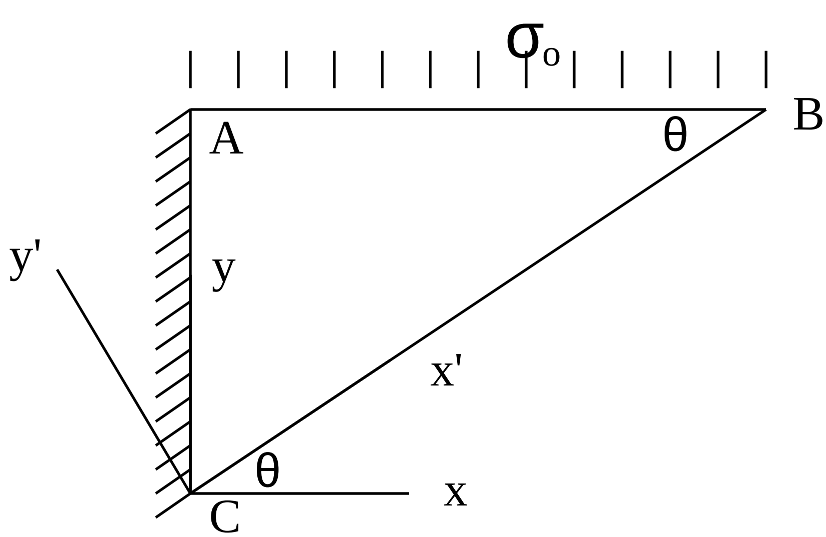
$$\sigma_{xx}|_{BC} = 0, \ \sigma_{xy}|_{BC} = -\tau_o(y).$$

Note that the shear stress sign is negative because of our sign convention (it is acting in the *negative* y-coordinate direction on a plane whose outward normal is in the *positive* x-coordinate direction.

Boundary CD: Again, displacements are not constrained and the applied boundary stresses are:

$$\sigma_{yy}|_{CD} = \sigma_o, \ \sigma_{yx}|_{CD} = 0$$

<u>Example 2:</u> Consider the two-dimensional problem shown in the figure.



Boundary AC: $u_x|_{AC} = 0$, $u_y|_{AC} = 0$.

Boundary AB: $\sigma_{yy}|_{AB} = -\sigma_o$ (note sign convention!), $\sigma_{yx}|_{AB} = 0$

Boundary BC: Here, again, the displacements are not constrained. So we have to prescribe the applied stresses. There are no normal or shear stresses applied across this boundary. But how are we to specify this? It is best to use a rotated x'y' frame that is aligned with this boundary as shown. Then with respect to this frame, we have:

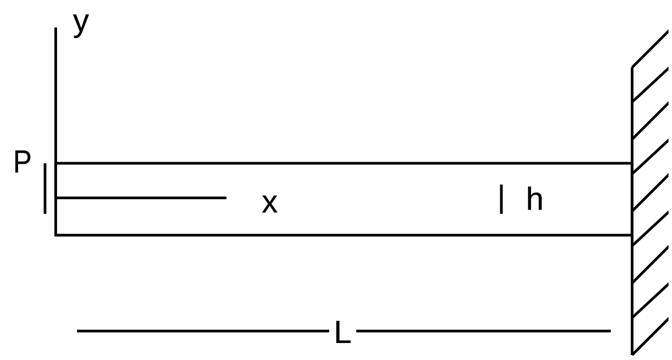
$$\sigma_{y'y'}|_{BC} = 0,\ \sigma_{y'x'}|_{BC} = 0.$$

We however do not want to carry along a whole bunch of different coordinate frames, and so we recast the boundary condition in terms the original xy frame using the stress transformation relations. Thus:

$$\sigma_{y'y'}\Big|_{BC} = \left[\sigma_{xx}\sin^2\theta + \sigma_{yy}\cos^2\theta - 2\sigma_{xy}\sin\theta\cos\theta\right]\Big|_{BC} = 0\,,$$

$$\sigma_{y'x'}\Big|_{BC} = \left[\sigma_{xy}(\cos^2\theta - \sin^2\theta) + (\sigma_{yy} - \sigma_{xx})\sin\theta\cos\theta\right]\Big|_{BC} = 0$$

Example 3: A rectangular thin plate of length L, height h, and thickness b, is fixed at x=L and loaded by a concentrated force P at the point x=0, y=0 as shown in the figure.



The plate is under plane stress conditions and it is suggested that the solution to this problem is:

$$\sigma_{xx} = Axy;\quad \sigma_{yy} = 0;\quad \sigma_{xy} = B(h^2 - 4y^2)$$

where the constants $A = 8B = 12\,P/(b\,h^3)$.

Clearly, the boundary conditions on the plane x = 0 are not satisfied since, according to the solution for the stresses given above, the loading on this boundary must be a parabolically distributed shear stress and not a concentrated load. That is, the given solution is in error at this boundary. However, we observe that the boundary conditions are satisfied in an average sense in that:

the net force in the x-direction: $\int_A \sigma_{xx}\,dA = 0$

the net force in the y-direction: $\int_A \sigma_{xy}\,dA = P$ the applied load.

It turns out that if the boundary conditions are satisfied in the above average sense, then the resulting solutions (even though in error near the boundary) are for the most part reasonably accurate sufficiently far away from the boundary. This was first conjectured by a guy called St. Venant and we are all eternally indebted to him!

Summary of Specification of Boundary Conditions:

We are now in a position to draw some conclusions regarding specification of boundary conditions.

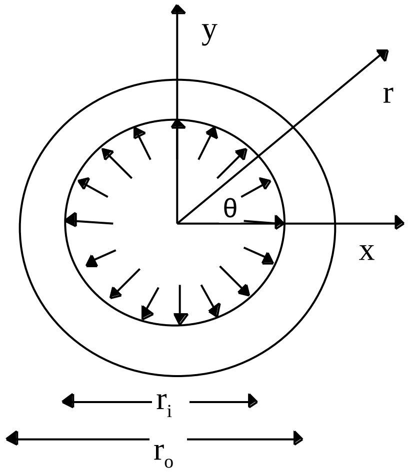
(i) The boundaries of a body may have either applied loads or prescribed displacement constraints. The variations of the displacements, strains and stresses within the body are solved using the elasticity equations while insisting on consistency of the solution with the boundary conditions.

(ii) We may not specify both the displacements and the loads in the same direction at the same point. If either one is prescribed, the other will follow from the solution to the problem. (This is similar to saying that we cannot prescribe both the stretch and the force on a given spring.)

(iii) Where a boundary of a body is not aligned with the chosen coordinate frame, it is convenient to use a new rotated coordinate frame and specify the boundary conditions in terms of the rotated frame. Then we need to transform the resulting boundary conditions back to the chosen coordinate frame.

(iv) Sometimes boundary conditions cannot be imposed in an exact sense. However, we may be able to get away with using "average" boundary conditions. The solution will be in error near the boundary, but will be okay for the most part elsewhere according to St. Venant.

EXAMPLE PROBLEM: PIPE UNDER PRESSURE



Consider a very long hollow pipe with internal radius 'r_i' and outer radius 'r_o'. If the pipe is under internal pressure 'p_i', what are the stresses in the pipe?

Since the pipe is given to be very long (in the z-direction in the figure), and since it is loaded only in the xy-plane, we expect that plane strain approximation is applicable to this situation.

Since we have circular boundaries, cylindrical polar coordinates are called for. With respect to the r,θ-coordinate located at the center of the pipe, the boundary conditions are:

$$\sigma_{rr}(r = r_i) = -p_i; \quad \sigma_{r\theta}(r = r_i) = 0;$$
$$\sigma_{rr}(r = r_o) = 0; \quad \sigma_{r\theta}(r = r_o) = 0; \qquad (*)$$

The Airy stress function applicable to this problem can be shown to be (can be looked up) (can be derived):

$$\Phi(r,\theta) = A \log r + Cr^2 \qquad (**)$$

Note that it is independent of 'θ'! This is not really surprising because the problem is *axisymmetric* - every radial line looks pretty much like any other radial line.
The constants A and C need to be determined from the boundary conditions (*).
From the relation between the Airy stress function and the stresses, we have:

$$\sigma_{rr} = \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} = \frac{A}{r^2} + 2C$$

$$\sigma_{\theta\theta} = \frac{\partial^2 \Phi}{\partial r^2} = -\frac{A}{r^2} + 2C$$

$$\sigma_{r\theta} = \frac{1}{r^2}\frac{\partial \Phi}{\partial \theta} - \frac{1}{r}\frac{\partial^2 \Phi}{\partial r \partial \theta} = 0 .$$

Using the above in the boundary conditions (*), we find:

$$\sigma_{rr}(r = r_i) = \frac{A}{r_i^2} + 2C = -p_i$$
$$\sigma_{rr}(r = r_o) = \frac{A}{r_o^2} + 2C = 0 \qquad (\dagger\dagger)$$

Solving (††), we can obtain the constants A and C for this problem:

$$A = -\left(\frac{r_i^2 r_o^2}{r_o^2 - r_i^2}\right)p_i; \quad 2C = \left(\frac{r_i^2}{r_o^2 - r_i^2}\right)p_i,$$

and the stresses in the pipe are therefore given by:

$$\sigma_{rr}(r) = \left\{\frac{r_i^2}{r_o^2 - r_i^2}\right\}\left\{1 - \frac{r_o^2}{r^2}\right\}p_i$$

$$\sigma_{\theta\theta}(r) = \left\{\frac{r_i^2}{r_o^2 - r_i^2}\right\}\left\{1 + \frac{r_o^2}{r^2}\right\}p_i$$

$$\sigma_{r\theta}(r) = 0.$$

Exercises:

(i) Try figuring out the principal stresses at any given point in the pipe, and assuming that the material of the pipe fails in tension, figure out where and along what direction failure is likely to occur. Do the same for a pipe that is expected to fail in shear.

(ii) How would you handle the problem of a pipe that is under external pressure p_o ?

(iii) From earlier courses on strength of materials, you know the solution to the problem of thin-walled pipes for which $r_o - r_i = t; \quad t << (r_i, \ r_o)$. Show that the above "exact" solution approximates to the strength of materials solution for a thin-walled pipe:

$$\sigma_{rr} \approx 0; \quad \sigma_{\theta\theta} \approx \frac{p_i r_i}{t}; \quad \sigma_{r\theta} = 0;$$