CHARACTERIZATION OF INTERNAL FORCES - STRESS

1.1 THE STRESS TENSOR

When external forces are applied to a solid body, the atoms or molecules in the body may move apart a little bit from each other. The reason the atoms hopefully do not completely come apart (if the load is sufficiently small) is because they resist the applied external forces by developing *internal* forces until equilibrium is achieved. If the internal forces cannot resist the external forces, the body breaks or fractures. Though internal forces are due to the atoms or molecules inside a body, it is too complex to study mechanical deformation from the atomistic point of view (even though some people make a living doing so). Therefore, we adopt the first of our many approximations, namely the so-called *continuum* assumption. Under this assumption we can forget about the details of the atomic structure of the solid, and instead treat the solid as if it were one continuous thing, whatever that means. In essence, our theory will hold only for length scales which are much larger than atomic distances.

Now, consider a cylindrical body that is subject to an external force F as shown in Fig 1a. Let us assume that the cylinder does not break apart under the action of the load.

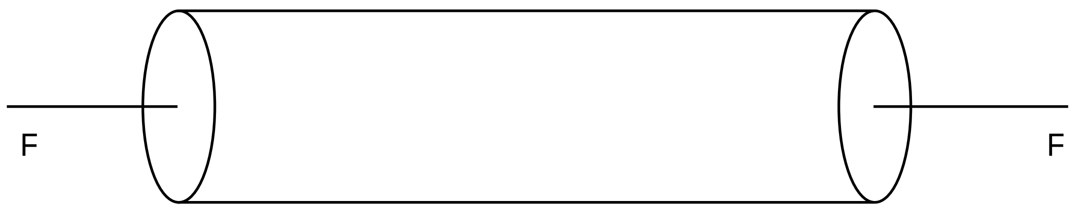


Figure 1a

To see how internal forces are created inside the cylinder to resist the applied force, let us make an imaginary cut through the body perpendicular to the cylinder axis and consider why the right half of the cylinder stays stuck to the left half. (Fig 1b).

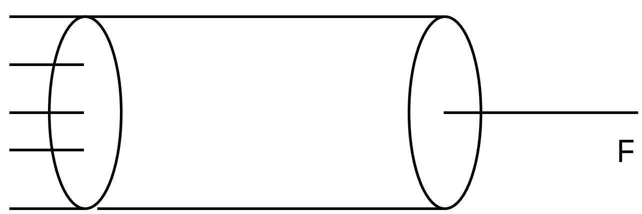


Figure 1b

Assuming everything is nice and uniform across the cut, we expect that all points on the cut surface will resist the applied force equally with an *intensity of force:*

$$s = \frac{F}{A} \quad , \tag{1}$$

where A is the cross-sectional area of the cylinder.

Let us now generalize this a bit. Suppose that the body were not nice and uniform. Maybe some termites have partially eaten into cylinder. In this case, we really have no right to expect that all points on a surface will resist the applied force equally. Some points may resist more and some less than others. So we generalize our intensity of force concept as follows. We take a small region ΔA across our cut surface and somehow find ΔF to be the resisting force over this area. Then the intensity of force over this area can be written as:

$$s = \frac{\Delta F}{\Delta A} . \qquad (2)$$

We then realize that if can make the small region ΔA smaller and smaller and find the corresponding internal force ΔF we can *localize* the definition of intensity of force at a point to:

$$s = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A} . \qquad (3)$$

Note that when I say that we shrink the area to zero, I do not really mean zero, because then I will be dealing with atoms and like creatures which we wish to avoid. What I mean by a point therefore is some region that is very small compared to the length scale I am interested in.

One other thing. Since forces are vectors that have both magnitude and direction, I should more properly write

$$\vec{s} = \lim_{\Delta A \to 0} \frac{\overrightarrow{\Delta F}}{\Delta A} , \qquad (4)$$

where now the intensity of force at a point is really a vector.

Looks like we are beginning to get a handle on characterizing internal forces with this intensity of force concept. Are we really, though? Consider, again, the nice and uniform cylinder we had in our first example before the termites put some holes in it. Now make an imaginary cut that is not perpendicular to the axis but is inclined at, say, 45^{o} to it (see Fig 2).

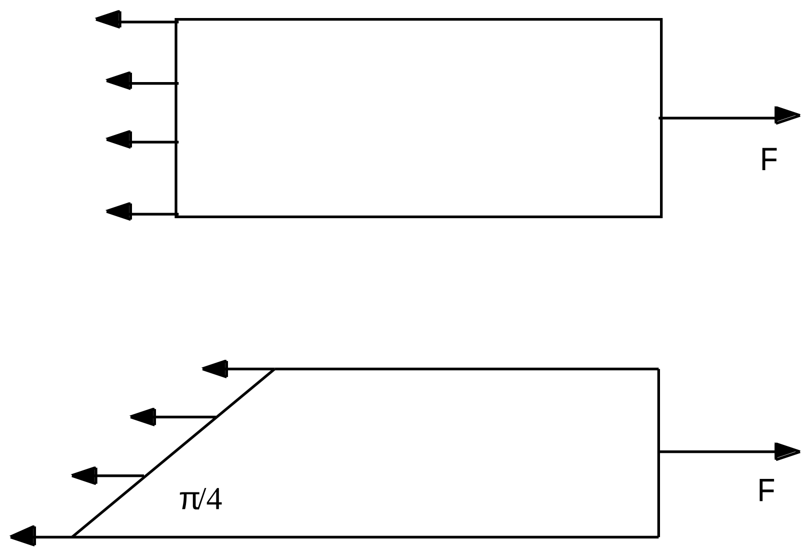


Figure 2

Now once again assuming that everything is nice and uniform over the cut area, what is the intensity of force in this case? Note that the area of the cut surface is now $A\sqrt{2}$ and hence the magnitude of the intensity of force is

$$s = \frac{F}{A\sqrt{2}} , \qquad (5)$$

and not what we had earlier. This looks like a serious set back in our effort to characterize internal forces! It seems that the way we have defined our intensity of force at a point depends on the way we make our imaginary cuts.

Since there are an infinity of planes that can cut through a point, we may have defined a quantity that is not really useful! Not to worry! Here is where we call upon **Cauchy's theorem** to come to our rescue. It turns out that Cauchy (mid 1800's) has proved this theorem that says that we need to consider at each point only three planes that are mutually perpendicular. The three intensity of force vectors associated with these three planes are sufficient to completely characterize the state of internal force at a point. That is, if we make an imaginary cut along any other plane passing through this point, the intensity of force associated with that plane is completely determined in terms of these three intensity of force vectors. I am not going to prove this theorem to you, but for now will ask you to take it as true. Later, in a homework problem, you will develop the stress transformation relations using what is essentially a proof of Cauchy's theorem.

For now, let us see what Cauchy's theorem implies. Define a Cartesian coordinate system for reference. Through any point in a body, it makes sense to consider the three planes that are normal (perpendicular) to the axes of our reference frame as the three planes needed to define the three intensity of force vectors which we shall label $\vec{s}_x, \vec{s}_y, \vec{s}_z$, where the subscript denotes that the plane under consideration is normal to the x, y and z axes respectively (see Fig. 3).

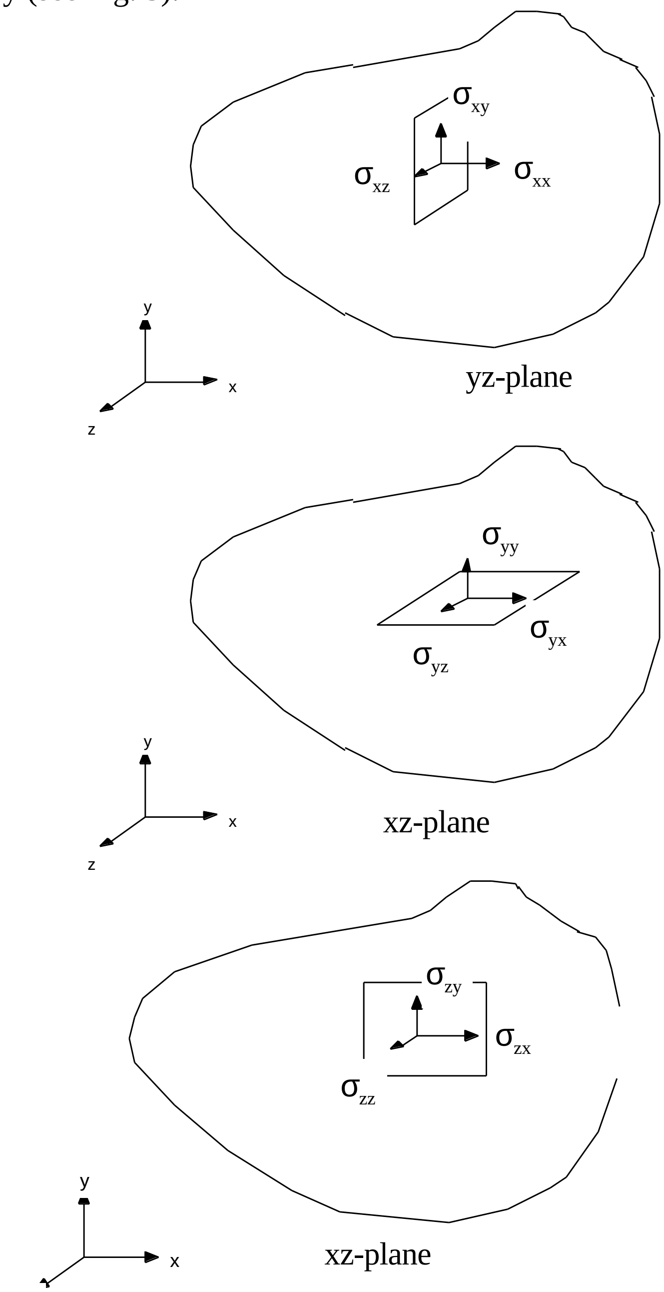


Figure: 3

Furthermore, we can resolve each of the three intensity of force vectors into their three components defined by the reference frame. We label these components as follows: the first subscript tells us the plane across which the stress component acts, and the second subscript tells us the direction. Thus we have nine quantities that define the state of internal forces at each point in a body. This collection of nine quantities, we shall call the **stress tensor** and we shall write them as:

$$\sigma = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}. \tag{6}$$

Each of these nine quantities, we shall call the **components** of the stress tensor or simply stress components with reference to our coordinate frame. The quantity that I have been so far calling the intensity of force, we shall call the **stress vector** or **traction** vector to distinguish it from the stress tensor.

Stress Element: Consider a cube of material scooped out from a body that is under some loads. If we draw all the stress components that act on the six faces of the element we get the three-dimensional stress-element below:

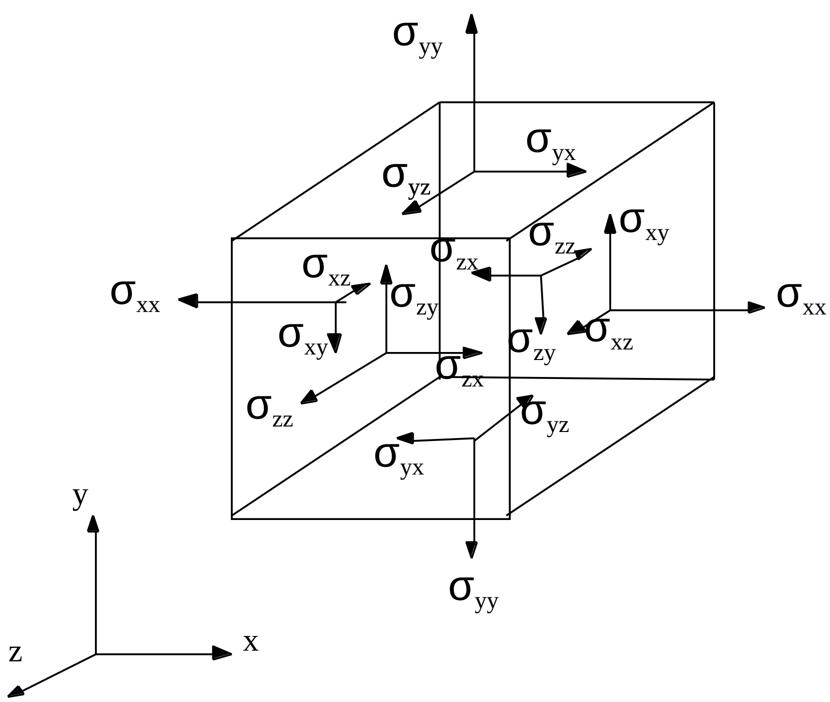


Figure: 4

Check that the notation for the stresses is consistent on all the planes in Fig 4. You may have noticed that I have sneaked in a sign convention for stresses in that I have drawn the stress component directions along the negative coordinate directions on some of the sides of the element. This is needed because, in the stress element shown, the normal stresses on any two opposite faces have the same effect when both are pointing outward from the body (causes the element to expand) or when both are pointing inward (causes the element to compress). Similarly for the shear stresses, but now in terms of direction of twisting of the element. Hence the **sign convention**:

> a stress component is treated as positive if both its direction as well as the outward (going away from the element under consideration) normal of the face its acting on are along coordinate directions of the same sign.

Can you see that this convention is consistent (as it should be) with Newton's third law by considering two neighboring stress elements?

Plane Stress: For certain thin plate like structures where the loading is in the plane of the plate (say in the xy plane) and nothing varies through the thickness of the structure, the only significant non-zero stress components are σ_{xx} , σ_{xy}, σ_{yx} , and σ_{yy} ; and furthermore these are essentially uniform through the thickness z-direction. Such a state of stress is called <u>plane</u> <u>stress</u>. Plane stress is a useful two-dimensionalization that is appropriate in some cases, as we will see later on.

1.2. EQUATIONS OF MOTION / EQUILIBRIUM

We have seen that the stress state in a body can vary through the body. However, this variation is subject to certain physical restrictions arising from Newton's laws of motion/equilibrium. Let us consider the case of a body that is either under static equilibrium or is in motion under the action of applied forces and constraints.

- For simplicity, to start with consider a two-dimensional body in a state of plane stress. We will therefore assume that the body is of unit thickness.

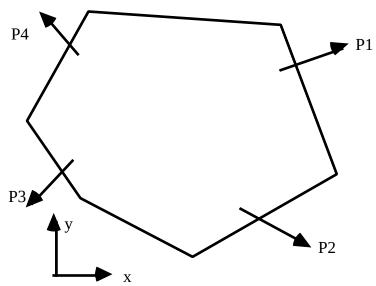


Figure 5

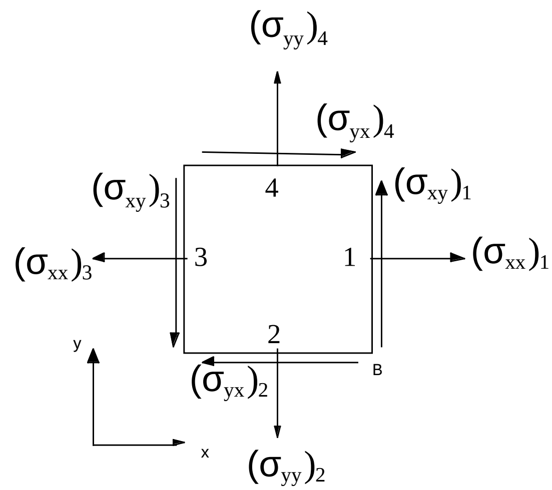


Figure 6

- Consider a rectangular stress element of sides Δx, Δy centered on any point (x,y) in the body. Let points 1, 2, 3, & 4 represent the centers of the sides of the element. Let us consider the free-body diagram of this element.

- Approximate the stresses acting on each face of the element by the value of the stress at the center of the face. Therefore, the total force acting on each face is the value of the stress at the center point of the face times the area of the face. Note that this is only an approximation which, however, will become exact as we shrink the element smaller and smaller.

- Note that the stresses on the various faces are in general different. That is $(\sigma_{xx})_1$ acting on face 1 is not necessarily equal to $(\sigma_{xx})_3$ et cetera as long as Δx, Δy are finite.

- Let us model gravity and magnetic forces that act throughout the volume of the body by means of body force densities (force/volume) f_x and f_y in the x- and y-directions respectively. The body force densities can also in general vary from point to point in a body (for instance if the mass density varies). However, we shall approximate these in the small element under consideration by their values at the center of the element. Therefore, the total body force in the element is the body force density at the center of the element times the volume of the element.

- Let a_x and a_y be the acceleration of the center point of the elements, and let α be the angular acceleration of the element about the z-axis. Again note that the accelerations can vary throughout the body (think of a fluid for instance). We will approximate the inertial forces on the small element under consideration by the mass of the element times the acceleration of the center of the element.

- Now let us apply Newton's laws of motion (equilibrium) to this stress element.

(i) First, consider the balance of moment due to all the forces acting on the element taken about the center point: $\sum M_{(o)} = I_z \alpha$:

{stress}.{area}.{moment arm} = {mass moment of inertia about center}.{angular accel}

$$\left\{(\sigma_{yx})_4 + (\sigma_{yx})_2\right\}\{\Delta x.1\}\{\Delta y/2\}$$

$$\left\{(\sigma_{xy})_3 + (\sigma_{xy})_1\right\}\{\Delta y.1\}\{\Delta x/2\} \qquad (7)$$

$$= \frac{1}{12}\{\rho.\Delta x.\Delta y.1\}\left\{(\Delta x)^2 + (\Delta y)^2\right\}\alpha$$

Divide above by Δx, Δy, and shrink element to zero to get:

$$\boxed{\sigma_{xy} = \sigma_{yx}}, \qquad (8)$$

where the points 1, 2, 3, and 4 all coalesce to the same point upon shrinking the element, and so we do not have to retain the point tags.

(ii) Next consider force balance along the x-direction: $\sum F_x = ma_x$

{stress}.{area} = {mass}.{linear accel}

$$\{(\sigma_{xx})_1 - (\sigma_{xx})_3\}.\{\Delta y.1\} + \{(\sigma_{yx})_4 - (\sigma_{yx})_2\}.\{\Delta x.1\} + f_x.\{\Delta x.\Delta y.1\} = \rho\{\Delta x.\Delta y.1\}.a_x \quad (9)$$

Once again, dividing by Δx, Δy, and shrinking the element to zero we get:

$$\lim_{\Delta x \to 0} \left\{ \frac{(\sigma_{xx})_1 - (\sigma_{xx})_3}{\Delta x} \right\} + \lim_{\Delta y \to 0} \left\{ \frac{(\sigma_{yx})_4 - (\sigma_{yx})_2}{\Delta y} \right\} + f_x = \rho\, a_x. \quad (10)$$

Recalling the definition of derivatives of functions, the above becomes:

$$\boxed{\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + f_x = \rho a_x}\,. \quad (11)$$

(iii) Finally, considering force balance in the y-direction, $\sum F_y = ma_y$, and repeating the steps in (ii) above, we get:

$$\boxed{\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + f_y = \rho a_y}\,. \quad (12)$$

- The generalization of the above to arbitrary three-dimensional states of stress is very straight forward. You can start with a general three dimensional parallelopiped stress element, and repeat the process above. Or else, you can stare hard at the three boxed equations above, and guess that the result must be:

$$\boxed{\begin{aligned} \sigma_{xy} &= \sigma_{yx} \\ \sigma_{yz} &= \sigma_{zy} \\ \sigma_{xz} &= \sigma_{zx} \end{aligned}}$$

from the three moment balance equations, and

$$\boxed{\begin{aligned} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + f_x &= \rho a_x \\ \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} + f_y &= \rho a_y \\ \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + f_z &= \rho a_z \end{aligned}}$$

from force balance along the x-, y-, and z-directions.

(13)

<u>Remarks</u>:

(i) The above are called the equations of motion in three-dimensions. The stress variation through a body cannot be arbitrary but must satisfy the above Newton's laws of motion for a continuum.

(ii) Note that we did not say anything about the nature of the material in deriving the equations of motion above. They are in fact valid for any material, elastic or inelastic solids; and even for fluids.

(iii) If a body is not accelerating, the right hand side of the above differential equations will be zero. In this case, they are called the equations of equilibrium.

1.3 STRESS TRANSFORMATION RELATIONS

Cauchy's theorem states that knowing the 9 stress components at a point with respect to one Cartesian reference frame (which defines three mutually perpendicular planes) we can obtain the stress components with respect to any other reference frame. To see how this works in general is algebraically a bit messy, so let us restrict attention to a body in a state of plane stress under static equilibrium with no applied body forces.

Suppose we know the stress components at any point P (see Figure 7) with respect to the xy-frame, i.e., we know σ_{xx}, σ_{yy}, $\sigma_{xy} = \sigma_{yx}$ at point P. We want to determine the intensity of internal force vector (more usually called the **stress vector** or **traction**) at point P across an arbitrary plane whose normal is rotated by an angle θ (taken positive counterclockwise) with respect to the x-axis. Equivalently, we want to determine the stress components at point P with respect to an x'y'-frame that is rotated with respect to the xy-frame.

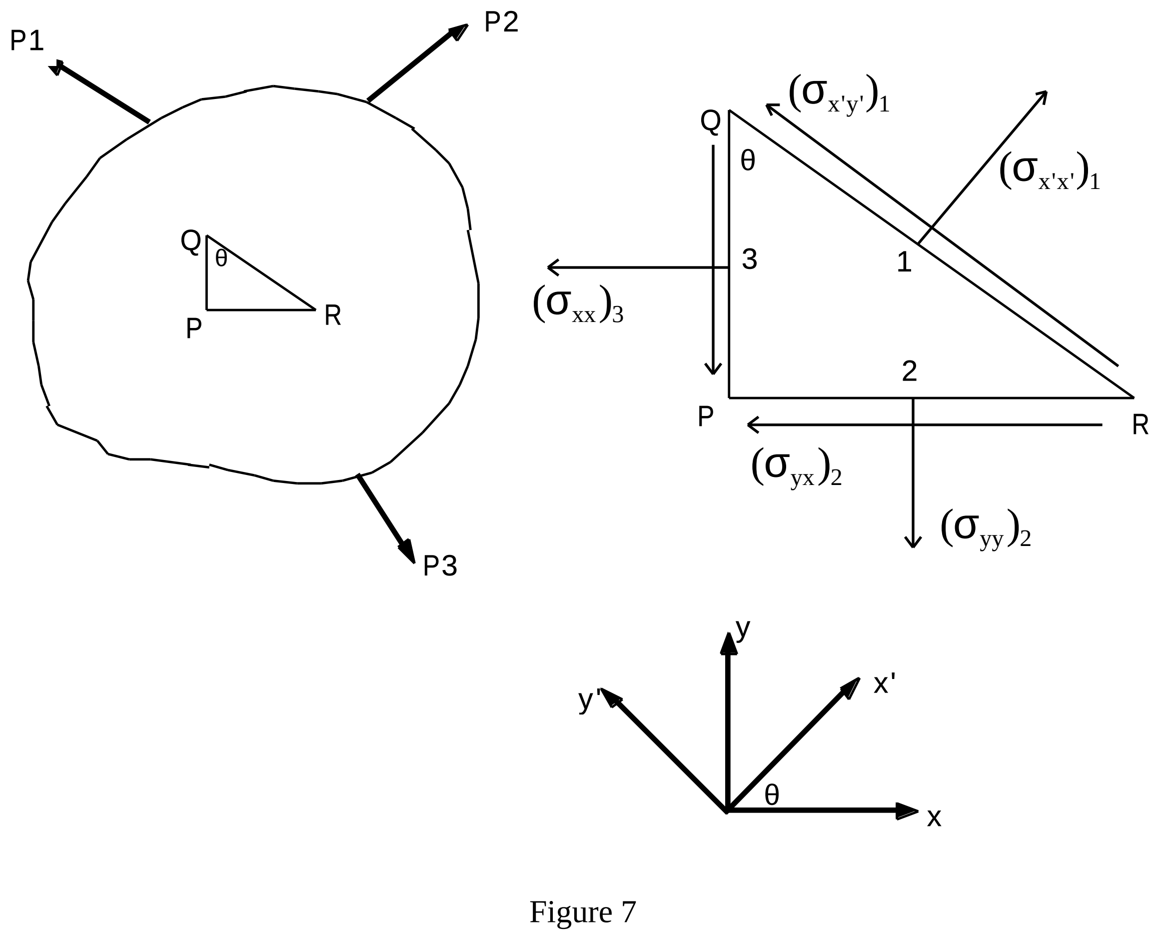


Figure 7

-Consider the triangular element PQR of sides PQ = Δy, PR = Δx, QR = Δl. Let its thickness be unity. Note that $\Delta x = \Delta l \sin\theta$ and $\Delta y = \Delta l \cos\theta$.

-Draw the stress components acting on the various faces of the element PQR. Approximate the stresses on each face of the triangle by their values at the center point. That is, on face PR the stress components can be approximated by their values at the center point 2, and will be $(\sigma_{yy})_2$ and $(\sigma_{yx})_2$ etc. Note that it is most convenient to represent the stress components on inclined face QR in terms of the x'y'-frame. This is because the outward normal to QR is along the x'-direction.

-For the element PQR to be under static equilibrium, Newton's laws must hold. Using our standard procedure, we consider the force equilibrium about the x-direction and then divide out by Δl and shrink Δl to zero to get:

$$\sigma_{x'x'} \cos\theta - \sigma_{x'y'} \sin\theta = \sigma_{xx} \cos\theta + \sigma_{xy} \sin\theta \qquad (*)$$

where the location tags have been dropped since all points 1,2 and 3 coalesce to the same point P.

-Repeat the above exercise for force equilibrium along the y-direction to get:

$$\sigma_{x'x'} \sin\theta + \sigma_{x'y'} \cos\theta = \sigma_{yy} \sin\theta + \sigma_{yx} \cos\theta \qquad (**)$$

-Solve the algebraic equations (*) and (**) to get

$$\begin{aligned} \sigma_{x'x'} &= \sigma_{xx} \cos^2\theta + \sigma_{yy} \sin^2\theta + 2\,\sigma_{xy} \sin\theta \cos\theta \\ \sigma_{x'y'} &= \sigma_{xy} \left(\cos^2\theta - \sin^2\theta\right) + (\sigma_{yy} - \sigma_{xx}) \sin\theta \cos\theta \end{aligned} \qquad (14)$$

- A similar procedure can be adopted to get:

$$\begin{aligned} \sigma_{y'y'} &= \sigma_{xx} \sin^2\theta + \sigma_{yy} \cos^2\theta - 2\,\sigma_{xy} \sin\theta \cos\theta \\ \sigma_{y'x'} &= \sigma_{x'y'} \end{aligned} \qquad (15)$$

Remarks:

(i) The stress transformation relations actually hold even if a body is subject to body forces and is under dynamic equilibrium. Can you see why?

(ii) What we have just developed is a non-rigorous ``proof'' of Cauchy's theorem. What are the fundamental principles or laws that have gone into your derivation? That is, on what principles does Cauchy's theorem rest?

(iii) The adventurous among you might want to try developing the stress transformation relations for the general three-dimensional case. You will have to consider a tetrahedral element in this case.

(iv) Using trigonometric formulas, express the above in terms of 2θ. The resulting expressions are simpler to use in some cases, and anyway it is nice to have them for your records.

Some Remarks on Vectors and Tensors:

(i) The above **stress transformation** equations in two dimensions can be written in matrix form as:

$$\begin{bmatrix} \sigma_{x'x'} & \sigma_{x'y'} \\ \sigma_{y'x'} & \sigma_{y'y'} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{16}$$

or more succinctly $\sigma' = \mathbf{T}\sigma\mathbf{T}^{\mathrm{T}}$ where **T** is called the transformation matrix. The above transformation relation is actually valid for any tensor quantity (In fact, any quantity that transforms according to (16) is *defined* to be a tensor).

(ii) The transformation matrix **T** is actually filled with something simple. Let $\{\mathbf{e_x}, \mathbf{e_y}\}$ and $\{\mathbf{e_{x'}}, \mathbf{e_{y'}}\}$ be unit vectors along the coordinate directions in the unprimed and primed coordinate frames. Then, it is easy to see that the transformation matrix is just:

$$\mathbf{T} = \begin{bmatrix} \mathbf{e_x}.\mathbf{e_{x'}} & \mathbf{e_y}.\mathbf{e_{x'}} \\ \mathbf{e_x}.\mathbf{e_{y'}} & \mathbf{e_y}.\mathbf{e_{y'}} \end{bmatrix} \tag{17}$$

where '.' represents scalar product of two vectors.

(iii) I urge you to show that the components of a 2 dimensional vector with respect to an xy and a rotated x'y' coordinate frame are in fact related through:

$$\mathbf{u}' = \mathbf{T}\mathbf{u} \tag{18}$$

where $\mathbf{u}^{\mathrm{T}} = \begin{bmatrix} u_x & u_y \end{bmatrix}$ and $\mathbf{u}'^{\mathrm{T}} = \begin{bmatrix} u_{x'} & u_{y'} \end{bmatrix}$. This follows very simply from geometry.

(iv) We can extend transformation to three dimensions, of course; that is, we can look at components of tensors (and vectors) in xyz and x'y'z' coordinate frames that are rotated in three dimensions to each other. One can obtain these rigorously in pretty much the same way that we did above for the stress tensor in two dimensions. Or else, we can guess that tensor transformation in three-d is going to look like: $\sigma' = \mathbf{T}\sigma\mathbf{T}^{\mathrm{T}}$ where **T** is now given by:

$$\mathbf{T} = \begin{bmatrix} \mathbf{e}_x.\mathbf{e}_{x'} & \mathbf{e}_y.\mathbf{e}_{x'} & \mathbf{e}_z.\mathbf{e}_{x'} \\ \mathbf{e}_x.\mathbf{e}_{y'} & \mathbf{e}_y.\mathbf{e}_{y'} & \mathbf{e}_z.\mathbf{e}_{y'} \\ \mathbf{e}_x.\mathbf{e}_{z'} & \mathbf{e}_y.\mathbf{e}_{z'} & \mathbf{e}_z.\mathbf{e}_{z'} \end{bmatrix} \tag{19}$$

and σ and σ' now have 9 components as in eqn. (6).

(v) What we have just seen are the rudiments of *tensor* algebra. In this algebra, the entities are:

- scalar quantities (such as mass, temperature) {scalars are more generally known as 0-tensors or tensors of rank 0 having $N^0 = 1$ component in N-dimensions}. They have the same value in any rotated coordinate frame.
- vector quantities (such as force, velocity) {vectors are more generally known as 1-tensors or tensors of rank 1 having $N^1 = N$ components in N-dimensions}. They have different components in different rotated coordinate frames, transforming according to: $\mathbf{u}' = \mathbf{T}\mathbf{u}$ where **T** is given in eqn. (19);
- tensor quantities (such as stress, and later we will see strain; and in electromagnetism the permeability) {tensors are more generally known as 2-tensors or tensors of rank 2 having N^2 components in N-dimensions}. 2-tensors also have different components in different rotated coordinate frames transforming according to: $\sigma' = \mathbf{T}\sigma\mathbf{T}^T$ where **T** is again given in eqn. (19).
- M-tensors are creatures that need N^M components to describe them, and their transformation all involve **T** in various combinations (examples of these include the elasticity 4-tensor)

Tensor algebra (and tensor calculus) is the language in which much of mechanics is now spoken. Unfortunately, we will not go into this in too much detail in this course.

1.4 PRINCIPAL STRESSES

For a certain class of brittle materials, a simplistic failure criterion might be that if the normal stress (or shear stress) at any point in the body *across any plane* reaches some critical value (depending on the material), the body will fracture. In this case, we will have to investigate the largest normal stresses (or shear stresses) at each point in the body after we have somehow obtained (through, say, a computer analysis of the structure) the stress components throughout the body with respect to some chosen coordinate frame.

To obtain the largest normal stresses, we can use the first of the stress transformation equations, and maximize with respect to orientation θ. It is convenient to first cast the stress transformation expression in terms of 2θ:

$$\sigma_{x'x'} = \frac{\sigma_{xx}+\sigma_{yy}}{2} + \frac{\sigma_{xx}-\sigma_{yy}}{2} \cos 2\theta + \sigma_{xy} \sin 2\theta$$

Our quest is to find that plane (given by angle θ) across which the above normal stress is maximum. Letting $\theta = \theta_p$ be the angle where this occurs, we must then have:

$$\left.\frac{d\sigma_{x'x'}}{d\theta}\right|_{\theta=\theta_p} = 0 \qquad \text{which gives} \qquad \tan 2\theta_p = \frac{2\sigma_{xy}}{\sigma_{xx} - \sigma_{yy}} \tag{20}$$

There are two solutions to the above which differ by 90^0. One of these maximizes the normal stress and the other minimizes the normal stress.

Remarks:

(i) The planes across which the normal stresses are maximum and minimum are called **principal planes** for the stress at that point, and the associated stresses are called **principal stresses**.

(ii) The principal stresses on the principal planes can be obtained by using the angle given by (16) in the stress transformation relations (14-15). We find:

the maximum principal stress $$\boxed{\left.\sigma_{x'x'}\right|_{\theta=\theta_p} \equiv \sigma_1 = \frac{\sigma_{xx} + \sigma_{yy}}{2} + \sqrt{\left[\frac{\sigma_{xx} - \sigma_{yy}}{2}\right]^2 + \sigma_{xy}^2}}$$

the minimum principal stress $$\boxed{\left.\sigma_{y'y'}\right|_{\theta=\theta_p} \equiv \sigma_2 = \frac{\sigma_{xx} + \sigma_{yy}}{2} - \sqrt{\left[\frac{\sigma_{xx} - \sigma_{yy}}{2}\right]^2 + \sigma_{xy}^2}} \tag{21}$$

and the shear stress $\boxed{\left.\sigma_{x'y'}\right|_{\theta=\theta_p} = 0}$.

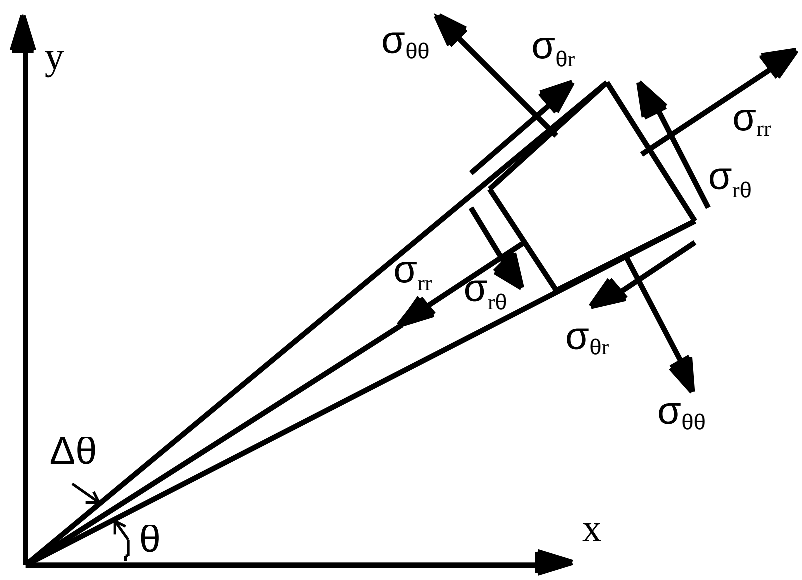
Note that the shear stress is zero across the principal planes.

(iii) A similar approach can be used to figure out planes across which the shear stress is a maximum (these will be different from the planes across which the normal stresses are maximum). This is left as an exercise to you.

1.5 STRESS COMPONENTS IN CYLINDRICAL POLAR COORDINATES

In some situations where the problem has cylindrical symmetry (such as a pipe under pressure), it is useful to consider cylindrical polar coordinates (rather than Cartesian) to define the three mutually perpendicular planes on which to obtain the stress components. Conceptually, everything we have done so far will carry over to polar coordinate case. You will explore some aspects of this in a homework problem.

Restricting ourselves to two-dimensional problems, consider a wedge element centered at (r, θ), and bounded in the radial (r) directions and the tangential (θ) directions as shown in the Figure.



Note that the stress components on the various sides of the wedge are labeled in a manner analogous to the scheme we used for Cartesian coordinates. The first subscript gives us the plane across which the stress acts, and the second gives us the direction.
By considering the static equilibrium (no dynamics and hence no acceleration of the body) of the wedge element under zero body forces , we find that

$$\frac{\partial\sigma_{rr}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{r\theta}}{\partial\theta}+\frac{\sigma_{rr}-\sigma_{\theta\theta}}{r}=0$$
$$\frac{\partial\sigma_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta}+\frac{2\sigma_{r\theta}}{r}=0 \qquad (22)$$
$$\sigma_{r\theta}=\sigma_{\theta r}$$

Convince yourself that this makes sense.