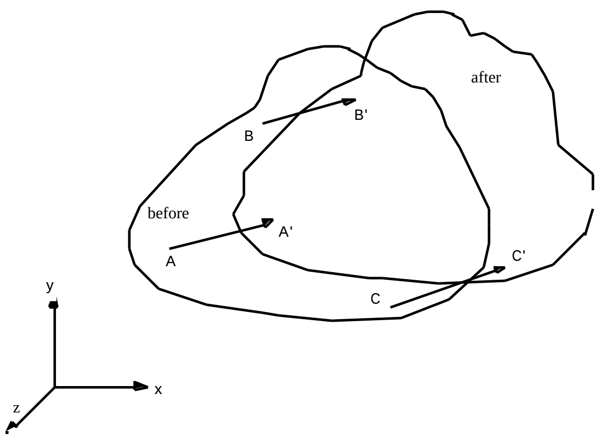
CHARACTERIZATION OF GEOMETRY OF DEFORMATION - DISPLACEMENTS & STRAINS

2.1 DISPLACEMENTS:

We now turn our attention to characterizing the geometry of deformation in a solid body. Consider a body that is subject to some external loads. The body deforms in shape from its original shape. We characterize this deformation by first keeping track of the way each point of the body in its undeformed/unloaded configuration moves to occupy a different point after deformation.



Obviously, point A moves to A', B to B' etc., and we can draw displacement vectors from A to A' and so on. With reference to a Cartesian coordinate frame, we can say that point A has been displaced by an amount u_x along the x-axis, u_y along the y-axis and u_z along the z-axis. That is: A(x,y,z) --> A'($x+u_x, y+u_y, z+u_z$). Similarly for points B, C etc.

Figure 1

Thus a set of three numbers { u_x ,u_y,u_z } at *each* point defines the **displacement** of that point upon application of external loads. Obviously { u_x ,u_y,u_z } can vary from point to point and so we should say displacements { u_x ,u_y,u_z }$|_A$ and so on. Instead of identifying points by an arbitrary label such as A or B, we can use the *original* coordinates of point A or B to identify the point itself. So we represent the displacements as $u_x(x,y,z)$, $u_y(x,y,z)$ and $u_z(x,y,z)$; and the way to interpret these quantities is as follows:

$$\left.\begin{matrix} u_x(x,y,z) \\ u_y(x,y,z) \\ u_z(x,y,z) \end{matrix}\right\} \quad \begin{matrix} \text{a point initially at (x,y,z)} \\ \text{displaces upon deformation to} \\ (x+u_x, y+u_y, z+u_z) \end{matrix}$$

Plane Deformation: For certain problems involving very thick bodies where nothing varies in the thickness direction and the loading is only in the plane of the body, a useful two-dimensional approximation is:

$u_z = 0$; $u_x(x,y)$; $u_y(x,y)$.

This means that no point gets displaced in the z-direction, and furthermore every plane parallel to the xy-plane deforms in the same manner. Turns out that plane deformation is a different kind of two-dimensional approximation than plane stress. More on this later.

2.2 STRAINS:

Since a body can displace (translate and/or rotate) without deforming (changing shape or size), and since it is only deformation that causes internal stresses, we need to extract that part of the displacement of a body that leads to deformation. That is, we need measures of change of size and shape of a body. These measures are called *strains*.

Now consider a two-dimensional body that has suffered a plane deformation.

- Let us draw two small line elements in the body before deformation as follows: AB of length Δx parallel to the x-direction; and AC of length Δy parallel to the y-direction.
- Upon deformation, these small line elements deform to A'B' and A'C' as shown in the figure. We will continue to treat the deformed line elements as straight lines, which is actually only an approximation that will be adequate provided the line elements are sufficiently small.

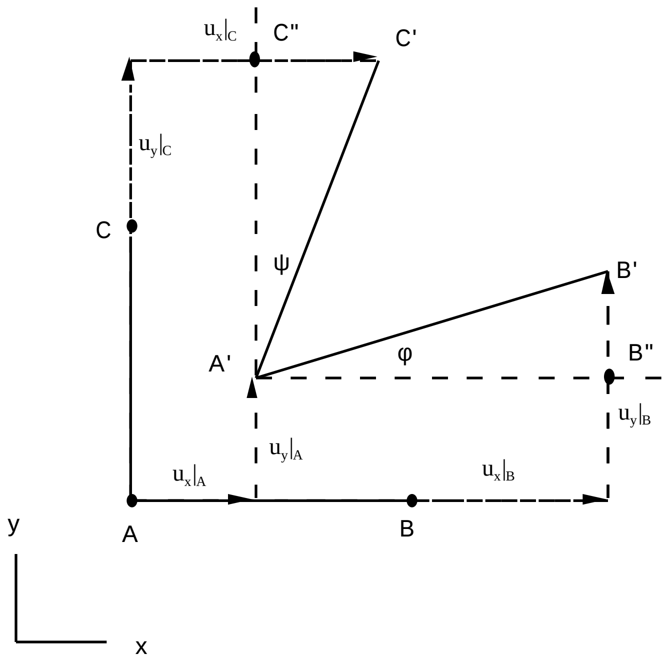


Figure 2

- Consider the change in length to original length of line element AB:

$$\frac{A'B - AB}{AB} \approx \frac{A'B'' - AB}{AB} = \frac{u_x|_B - u_x|_A}{\Delta x} \quad \text{provided } \varphi \text{ is small.}$$

Based on the above, we now define a **normal strain**:

$$\boxed{\varepsilon_{xx} = \lim_{\Delta x \to 0} \left\{ \frac{u_x|_B - u_x|_A}{\Delta x} \right\} = \frac{\partial u_x}{\partial x}} \qquad (1)$$

The above normal strain measure can be obtained at any point in the body and is a measure of the change in length to the original length of a line element initially along the x-direction.

- Consider now the change in length to original length of line element AC:

$$\frac{A'C'-AC}{AC} \approx \frac{A'C''-AC}{AC} = \frac{u_y|_B - u_y|_A}{\Delta y} \quad \text{provided } \psi \text{ is small.}$$

Based on the above, we can define another normal strain:

$$\boxed{\varepsilon_{yy} = \lim_{\Delta y \to 0} \left\{ \frac{u_y|_B - u_y|_A}{\Delta y} \right\} = \frac{\partial u_y}{\partial y}} \qquad (2)$$

Again, the above normal strain measure can be obtained at any point in the body and is a measure of the change in length to the original length of line elements initially along the y-direction.

- To characterize change of shape, we can obtain a measure of the change in *angle* between AC and AB which is $\phi + \psi$. Consider:

$$\phi \approx \tan\phi = \frac{B'B''}{A'B''} \approx \frac{u_y|_B - u_y|_A}{\Delta x \left\{ 1 + \frac{u_x|_B - u_x|_A}{\Delta x} \right\}} \xrightarrow{\Delta x \to 0} \frac{\partial u_y}{\partial x} \quad \text{provided } \varepsilon_{xx} \text{ is small.}$$

Similarly

$$\psi \approx \tan\psi = \frac{C'C''}{A'C''} \approx \frac{u_x|_C - u_x|_A}{\Delta y \left\{ 1 + \frac{u_y|_C - u_y|_A}{\Delta y} \right\}} \xrightarrow{\Delta y \to 0} \frac{\partial u_x}{\partial y} \quad \text{provided } \varepsilon_{yy} \text{ is small.}$$

We therefore define two **shear strains**:

$$\boxed{\varepsilon_{xy} = \varepsilon_{yx} = \frac{1}{2}(\phi + \psi) = \frac{1}{2}\left\{ \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right\}} \qquad (3)$$

where we have equitably distributed one-half the change in angle to each shear strain. These shear strains are therefore a measure of half the change in angle between two line elements that were initially along the x- and y-directions respectively.

- Generalizing the above strain measures to the three dimensional case, we can define three normal strains and six shearing strains as:

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} \qquad \varepsilon_{xy} = \varepsilon_{yx} = \frac{1}{2}\left\{\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right\}$$

$$\varepsilon_{yy} = \frac{\partial u_y}{\partial y} \qquad \varepsilon_{yz} = \varepsilon_{zy} = \frac{1}{2}\left\{\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}\right\} \qquad (4)$$

$$\varepsilon_{zz} = \frac{\partial u_z}{\partial z} \qquad \varepsilon_{xz} = \varepsilon_{zx} = \frac{1}{2}\left\{\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right\}$$

The three normal strains are a measure of change of size (length) of line elements along the appropriate coordinate directions, and the six (of which only three are distinct) shear strains are a measure of change of shape (angle) between pairs of line elements initially along the corresponding coordinate directions.

It turns out that these nine numbers are all that we need to characterize the state of strain at any point in the body. In fact, knowledge of the elongation and shearing of line elements with respect to the xyz-directions, is sufficient to determine the elongation or shear in any other direction. (I am not going to prove this, but it follows directly from geometry. You can look it up in one of the references, and more on this in a homework problem). Actually, by now, you must have guessed that these nine numbers seem to be awfully similar to the nine components of the stress tensor. In fact we shall group them as:

$$\varepsilon = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} \qquad (5)$$

and call it the **strain tensor**, and call each of the nine strain measures the components of the strain tensor or strain components with respect to the reference frame.

Remarks: In our characterization of the geometry of deformation, we have assumed that the deformation is small, or more correctly, that the deformation derivatives are small. If the application of external loads to a body leads to large deformation (strains), we should not use this theory, but develop a more accurate one. For most engineering solid structures however, small strain approximation is pretty good.

2.3 STRAIN TRANSFORMATION RELATIONS:

The relations that connect the strain components in one reference frame to any other frame are analogous to the stress transformation equations. By considering line elements that were initially along x, y and z coordinate directions, we defined the nine components of the strain tensor (eq. 5). What do we do about line elements that are at an angle to the xyz coordinate directions? Equivalently, what if we had considered a different x'y'z' coordinate frame? In this case, we would have had another set of nine strain components:

$$\varepsilon' = \begin{bmatrix} \varepsilon_{x'x'} & \varepsilon_{x'y'} & \varepsilon_{x'z'} \\ \varepsilon_{y'x'} & \varepsilon_{y'y'} & \varepsilon_{y'z'} \\ \varepsilon_{z'x'} & \varepsilon_{z'y'} & \varepsilon_{z'z'} \end{bmatrix}$$

What do you think these measure? Make sure that you answer this question before reading further

As luck (actually it is *not* luck, but geometry!) would have it, it turns out that strain is a 2-tensor. I am not going to prove this,but will ask you to trust me on this one.Therefore, knowing the strain components with respect to one coordinate frame, we can obtain the strain components with respect to any other coordinate frame using:

$$\varepsilon' = \mathbf{T}\varepsilon\mathbf{T}^T$$

where **T** is the same transformation matrix we saw in the previous chapter.

Aside on engineering shear strains: You might wonder why we introduced the factor half in the definition of the shear strain. Actually there is something called *engineering* shear strain $\gamma_{xy} = 2\varepsilon_{xy}$, et cetera that does not contain what you might consider as a nuisance factor of half. We will use the engineering shear strains extensively later on when we look at numerical solutions. However, it turns out that the nuisance factor of half is necessary in order to make strain a 2-tensor. That is, (5) behaves like a tensor (ie transforms like a tensor) only with the half in the definition of the shear strains. We just have to live with it for now. And you have to remember that engineering shear strains are twice the tensor definition of shear strains.

2.4 COMPATIBILITY EQUATIONS:

Paralleling our development of the concept of stress, we shall now inquire whether there are any physical restrictions that apply to the strain components. In the case of stresses, we had Newton's laws to satisfy. Strains, being a geometric measure of deformation, must obey some geometric constraints. Physically, we shall now require that a body not break apart but stay together upon deformation (this is just a *restriction* on allowable deformation, not a physical law - bodies do break!). Also, points A and B which were initially different should not merge into one point after deformation and vice versa. Mathematically, this requires that all points on the body before deformation map to points after deformation in a one-to-one and sufficiently smooth manner. This leads to the so-called equations of compatibility.

Note that in the two-dimensional case, we defined three strain quantities from only two displacement quantities. We get the feeling that they must somehow be related if the displacements of the body are nice and smooth (no breaks in the body, and two points do not become one after deformation etc). Indeed, this relation can be obtained from the strain displacement equations as follows:

Take two derivatives of ε_{xx} with respect to 'y': $\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} = \frac{\partial^3 u_x}{\partial y^2 \partial x}$

Take two derivatives of ε_{yy} with respect to 'x': $\frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = \frac{\partial^3 u_y}{\partial x^2 \partial y}$

Take two derivatives of ε_{xy} -- once with respect to 'x' and once with respect to 'y':

$$\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} = \frac{1}{2}\left\{ \frac{\partial^3 u_x}{\partial y^2 \partial x} + \frac{\partial^3 u_y}{\partial x^2 \partial y} \right\}$$

Stare hard at the three expressions above and note that:

$$\boxed{\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y}} \qquad (6)$$

What the above says is that we cannot have any arbitrary strain distribution in a body, if we want to ensure that the deformed body does not have holes in it or has several points merging into one etc. The strains must be such that they satisfy the compatibility eqn. (6).

Remarks: (i) So, what do my physical restrictions on allowable deformations have to do with the above algebraic blitzkreig? The requirement that the deformation be smooth etc is precisely what allowed us to take the required number of derivatives in the above.

(ii) For the general three-dimensional case, one has a total of six equations of compatibility that the strain components must satisfy.

2.5 DISPLACEMENTS AND STRAINS IN POLAR COORDINATES:

There are times when it is advantageous to use cylindrical polar coordinates to keep track of displacements and strains. In order to characterize the geometry of deformation, we now keep track of *displacements* of points along the radial r and tangential θ directions as shown in figure (in two dimensions only). Thus we have the displacement functions: $u_r(r,\theta)$, $u_\theta(r,\theta)$ that keep track of particle displacement upon deformation.

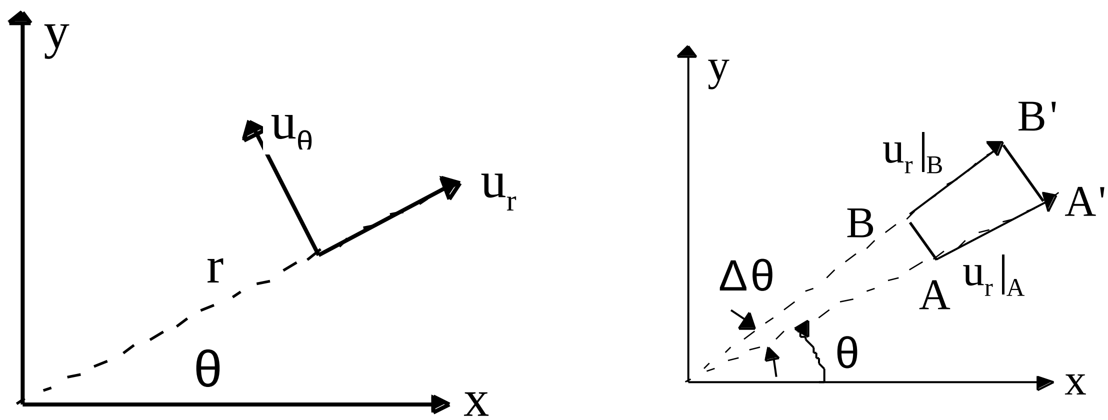


Figure 3

By considering line elements that are along the radial and tangential directions before deformation, and monitoring their new positions upon deformation, we can define strain components in cylindrical polar coordinates:

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}$$

$$\varepsilon_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}$$

$$\varepsilon_{r\theta} = \varepsilon_{\theta r} = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r}\right)$$

The first two are radial and tangential normal strains, and the last is a shear strain. Their meaning should be rather obvious to you.

Remark: One thing you might want to ponder about is this: If a body undergoes purely radial deformation (ie. every point moves radially out or in, and so u_r exists but $u_\theta=0$ everywhere as shown in the figure), then is the tangential normal strain $\varepsilon_{\theta\theta}$ zero? Why or why not?