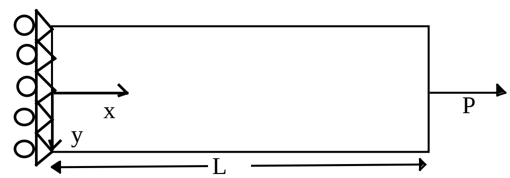
6.7 RAYLEIGH-RITZ APPROACH TO STRETCHING OF RODS:

Uniform Rods:

Consider a uniform rod of length L, cross-sectional area A, made of material with Young's modulus E, and which is pinned at one end and carries a load P at the other as shown. We know the "exact" solution to this problem already from our earlier adventures in theory of elasticity.



$$\sigma_{xx} = P/A$$

$$\varepsilon_{xx} = \sigma_{xx}/E = P/AE$$

$$u_x(x) = \int \varepsilon_{xx} dx = \frac{P}{AE} x$$

But let us see if the Rayleigh-Ritz method delivers the goods here. We need to create a bag of candidate displacements that must satisfy the geometric boundary conditions:

$$u_x(0) = 0. \qquad \text{(gbc)}$$

Clearly, since the exact solution is linear in 'x', we would be best off seeking such a form. But I am going to intentionally choose a quadratic dependence which still satisfies the gbc as follows:

$$u_x(x) = a_1\left(\frac{x}{L}\right) + a_2\left(\frac{x}{L}\right)^2 \quad \text{for } 0 \le x \le L$$

If the RR (and more importantly the PMPE) machinery is any good at all, we know what should happen.

Calculating the stored energy for a rod, we have:

$$U = \int_0^L \frac{EA}{2}\left\{\frac{du_x}{dx}\right\}^2 dx$$

$$= \int_0^L \frac{EA}{2}\left\{\frac{a_1}{L} + \frac{2a_2 x}{L^2}\right\}^2 dx$$

$$= \frac{EA}{2L}\left\{a_1^2 + \frac{4}{3}a_2^2 + 2a_1 a_2\right\}$$

and the virtual work term is:

$$W = Pu_x(L) = P\{a_1 + a_2\}$$

and so the potential energy functional is:

$$\Pi = \frac{EA}{2L}\left\{a_1^2 + \frac{4}{3}a_2^2 + 2a_1 a_2\right\} - P\{a_1 + a_2\}.$$

Minimizing the potential energy functional with respect to a_1 and a_2, we have:

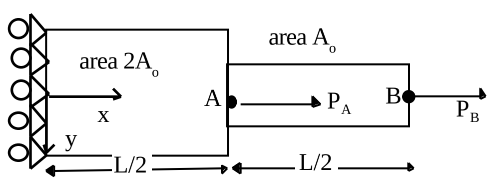
$$\frac{\partial \Pi}{\partial a_1} = 0 \qquad \frac{EA}{2L}\left\{2a_1 + 2a_2\right\} - P = 0$$

$$\frac{\partial \Pi}{\partial a_2} = 0 \qquad \frac{EA}{2L}\left\{2a_1 + \frac{8}{3}a_2\right\} - P = 0$$

which give us the necessary equations for us to solve for a_1 and a_2. Sure enough we find that: $a_1 = P/AE$ and $a_2 = 0$. That is, we recover the *exact* solution from the bag using PMPE *since our RR-bag already contained the exact solution.* With this result, your confidence in PMPE and RR should be improving!

Remarks: The above is obviously a lot more complicated than just using our earlier approach to this problem via Newton's laws of equilibrium . The real power of the RR-method will be more apparent when we consider rods of more complicated shapes - stepped or continuously-varying rods. In the above, all we have had to do is choose an appropriate bag, and then make sure that in calculating the stored energy, we are aware that the cross-sectional area now varies, that is A(x), and so the integration is a bit more involved.

<u>Towards finite elements - A Stepped Rod:</u>



Consider a 2-stepped rod made of the same material of Young's modulus E as shown in the figure. We can seek an approximate solution using RR.

(i) Bag of candidate displacements: We no longer expect that a linear function in 'x' is a good solution over the entire rod, so we maybe tempted to try a quadratic or higher order dependence as long as they satisfy the gbc: $u_x(0) = 0$. But I am going to try a different approach now which involves using a *multi*-part RR function:

$$u_x(x) = \begin{cases} 2u_A \, x/L & \text{for } 0 \le x \le L/2 \\ (2u_A - u_B) + 2(u_B - u_A)\, x/L & \text{for } L/2 \le x \le L \end{cases}.$$

Check that the above satisfies the gbc at x=0; that u_A and u_B are the displacements at A and B respectively; and that the above provides a displacement that is continuous across x=L/2. What we have done is to use a linear RR-function for each uniform segment of the stepped rod. That is, we are no longer confining ourselves to a single RR-function that is valid throughout the structure. If you think about it a bit, it makes great sense to do so.

(ii) Compute the potential energy functional:

The stored energy is:

$$U = \int_0^L \frac{EA}{2}\left\{\frac{du_x}{dx}\right\}^2 dx$$
$$= \int_0^{L/2} \frac{E(2A_0)}{2}\left\{\frac{2u_A}{L}\right\}^2 dx + \int_{L/2}^{L} \frac{EA_0}{2}\left\{\frac{2(u_B-u_A)}{L}\right\}^2 dx$$
$$= \frac{EA_0}{L}\left\{3u_A^2 + u_B^2 - 2u_Au_B\right\}$$

and the virtual work is: $W = P_Au_A + P_Bu_B$.

The potential energy functional is therefore:

$$\Pi = \frac{EA_0}{L}\left\{3u_A^2 + u_B^2 - 2u_Au_B\right\} - \left\{P_Au_A + P_Bu_B\right\}$$

(iii) Minimizing the potential energy functional with respect to u_A and u_B leads to:

$$\frac{\partial \Pi}{\partial u_A} = 0 \qquad \frac{EA_o}{L}\left\{6u_A - 2u_B\right\} - P_A = 0$$
$$\frac{\partial \Pi}{\partial u_B} = 0 \qquad \frac{EA_o}{2L}\left\{-2u_A + 2u_B\right\} - P_B = 0$$

Let us re-write this in matrix form as:

$$\underbrace{\begin{bmatrix} 6EA_0/L & -2EA_0/L \\ -2EA_0/L & 2EA_0/L \end{bmatrix}}_{\text{stiffness matrix}} \underbrace{\begin{Bmatrix} u_A \\ u_B \end{Bmatrix}}_{\substack{\textit{displacement}\\ \textit{vector}}} = \underbrace{\begin{Bmatrix} P_A \\ P_B \end{Bmatrix}}_{\substack{\textit{load}\\ \textit{vector}}}.$$

The solution to this is: $\begin{Bmatrix} u_A \\ u_B \end{Bmatrix} = \begin{Bmatrix} \dfrac{PL}{4EA_0} \\ \dfrac{3PL}{4EA_0} \end{Bmatrix}$

Remarks: Multi-part Rayleigh-Ritz techniques lead directly to a popular computational technique called displacement-based finite element method (FEM). The matrix representation above is standard finite element notation. The stiffness matrix is denoted **K**, the displacement vector is called **d**, and the force or load vector is called **F**, where **Kd**=**F**. In FEM, the structure is assumed to be composed of "elements" (each of the segments in our example) connected only at the nodes (the pin, and points A and B in our example). The nodal displacements are the Rayleigh-Ritz parameters to be determined by minimizing the potential energy. The displacements inside each element are given by the assumed Rayleigh-Ritz functions, which in FEM terminology are called shape functions.

6.8 FEM FORMULATION IN ONE-DIMENSION:

Consider a system of rods of which we pull out element 'e' and lool at its free-body diagram. Let its nodes be labeled 'I' and 'J' and let their location be x_I and x_J. The nodal displacements are u_I and u_J. The cross-section of the rod is A_e, its length $L_e=x_J-x_I$, and its Young's modulus is E_e. Let applied forces F_I and F_J act on the nodes as shown.

- Seek a linear RR-displacement function ("shape function"):

$$u_x(x) = \alpha + \beta x \quad \text{for} \quad x_I \leq x \leq x_J \qquad \text{(a)}$$

Cast the unknowns α and β in terms of the nodal displacements using the fact that:

$$\begin{aligned} u_x(x_I) &= \alpha + \beta x_I = u_I \\ u_x(x_J) &= \alpha + \beta x_J = u_J \end{aligned}.$$

Solving for α and β , we can re-express (a) as:

$$u_x(x) = \begin{bmatrix} \frac{x_J - x}{L_e} & \frac{-x_I + x}{L_e} \end{bmatrix} \begin{Bmatrix} u_I \\ u_J \end{Bmatrix}.$$

In short-hand notation, this can be written as:

$$\boxed{\mathbf{u_e} = \mathbf{N_e d_e}} \qquad (*)$$

where the **displacement function** vector is $\mathbf{u_e} = \{u_x(x)\}$ and $\mathbf{d}_\mathbf{e}^T = \{u_I \quad u_J\}$ is the ***nodal displacement*** vector, and $\mathbf{N_e} = \begin{bmatrix} \frac{x_J - x}{L_e} & \frac{-x_I + x}{L_e} \end{bmatrix}$ is the **shape function** matrix.

- The strain is then given by $\varepsilon_{xx}(x) = \frac{\partial u_x(x)}{\partial x}$ which can be written in terms of the nodal displacement vector as

$$\boxed{\varepsilon_\mathbf{e} = \mathbf{B_e d_e}} \qquad (**)$$

where the strain vector is $\varepsilon_\mathbf{e}^T = \{\varepsilon_{xx}(x)\}$ and $\mathbf{B_e} = \begin{bmatrix} \frac{-1}{L_e} & \frac{1}{L_e} \end{bmatrix}$ is obtained using (*). $\mathbf{B}_e$ is called the kinematic or B-matrix for the element. You may be wondering why I am writing all this in matrix form, but hold on awhile.

- The stress can be obtained from the strain using the stress-strain relation:

$$\boxed{\sigma_e = \mathbf{D}_e \varepsilon_e}$$

where $\sigma_e = \{\sigma_{xx}\}$, and here $\mathbf{D}_e = \{E_e\}$ is the elasticity matrix in one dimensions.

- The strain energy stored in the rod element is:

$$U_e = \frac{1}{2}\int_V \varepsilon_e^T \mathbf{D}_e \varepsilon_e \; dV$$

$$= \frac{1}{2}\int_V \mathbf{d}_e^T \mathbf{B}_e^T \mathbf{D}_e \mathbf{B}_e \mathbf{d}_e \; dV \quad \text{expressing in terms of the nodal displacement vector}$$

$$= \frac{1}{2}\mathbf{d}_e^T \left\{ \int_V \mathbf{B}_e^T \mathbf{D}_e \mathbf{B}_e \; dV \right\} \mathbf{d}_e \quad \text{pulling out } \mathbf{d}_e \text{ since it is a constant vector}$$

The last step in the above is an explicit recognition of the fact that the nodal displacement vectors, being constants, can be pulled out of the integral over 'x' (note that the $\mathbf{B}_e$ matrix may in general depend on 'x'), leaving the term in curly brackets

$$\boxed{\mathbf{K}_e = \int_V \mathbf{B}_e^T \mathbf{D}_e \mathrm{B}_e \; dV} \quad (\dagger)$$

which is called the element **stiffness** matrix. Thus the strain energy stored in a rod element is:

$$\boxed{U_e = \frac{1}{2}\mathbf{d}_e^T \mathbf{K}_e \mathbf{d}_e} \qquad (***)$$

In this case of uniaxial stretching, the stiffness matrix evaluates to:

$$\mathbf{K}_e = \int_{x_I}^{x_J} \mathbf{B}_e^T \mathbf{D}_e \mathrm{B}_e \; A_e dx = \mathbf{B}_e^T \mathbf{D}_e \mathrm{B}_e A_e \underbrace{\int_{x_I}^{x_J} dx}_{=L_e}$$

$$= \frac{A_e E_e}{L_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Note that the stiffness matrix is symmetric.

The total strain energy stored in all the elements making up the system of rods is therefore:

$$U = \sum_{\text{all elements}} U_e = \sum_{\text{all elements}} \frac{1}{2}\mathbf{d}_e^T \mathbf{K}_e \mathbf{d}_e$$

- The virtual work term for the entire rod system is just the sum of (force on nodes)*(nodal displacements) for all the nodes: That is:

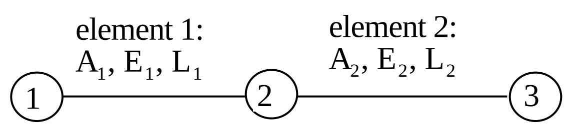
$$W = F_1 u_1 + F_2 u_2 + ... = \mathbf{d}^T \mathbf{F}$$

where $\mathbf{F}^T = \{F_1 \;\; F_2 \; F_3 \;\; F_4 \; ...\}$ is the **global load vector** consisting of all the applied forces at all the nodes of the system, and $\mathbf{d}^T = \{d_1 \;\; d_2 \;\; d_3 \;\; d_4 \; ...\}$ is the **global displacement vector** consisting of the displacements at all the nodes in the system. (Note that $\mathbf{d}_e$ contains only the two nodes that are attached to the element 'e'.)

- The potential energy functional is therefore:

$$\boxed{\Pi = \sum_{all \;\; elements} \frac{1}{2} \mathbf{d}_e^T \mathbf{K}_e \mathbf{d}_e - \mathbf{d}^T \mathbf{F}}$$

- The final step is to minimize the potential energy functional with respect to the nodal displacements (the RR-parameters) u_1, u_2, u_3,.... Before doing this, however, it turns out that it is useful to recast the strain energy stored in each element in terms of the global displacement vector $\mathbf{d}$ rather than $\mathbf{d}_e$. To see how we can achieve this, and also to make some of the ideas upto this point more concrete, let us consider a two-stepped rod. It can be treated as made of two elements with three nodes as shown.



The element stiffness matrices can immediately be written as:

$$\mathbf{K_1} = \frac{A_1 E_1}{L_1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} \\ k_{21}^{(1)} & k_{22}^{(1)} \end{bmatrix} \textit{(say)} ;$$

$$\mathbf{K_2} = \frac{A_2 E_2}{L_2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} k_{11}^{(2)} & k_{12}^{(2)} \\ k_{21}^{(2)} & k_{22}^{(2)} \end{bmatrix} \textit{(say)}. \qquad (\dagger\dagger)$$

The energy stored in rod elements '1' is: $U_1 = \frac{1}{2} \mathbf{d_1^T K_1 d_1}$ where $\mathbf{d_1^T} = \{u_1 \quad u_2\}$ since only nodes '1' and '2' are connected to element '1'. Expanding this, we have:

$$U_1 = \frac{1}{2} \mathbf{d_1^T K_1 d_1} = \frac{1}{2} \{u_1 \quad u_2\} \begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} \\ k_{21}^{(1)} & k_{22}^{(1)} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$= \frac{1}{2} \{ k_{11}^{(1)} u_1^2 + k_{12}^{(1)} u_1 u_2 + k_{21}^{(1)} u_1 u_2 + k_{22}^{(1)} u_2^2 \}$$

The energy stored in rod element '2' is in like manner: $U_2 = \frac{1}{2} \mathbf{d_2^T K_2 d_2}$ where $\mathbf{d_2^T} = \{u_2 \quad u_3\}$ since only nodes '2' and '3 are connected to element '2'.

We can rewrite the energy stored in each of the elements in terms of the *global* displacement vector which however includes displacements at all three nodes in this example, i.e

$\mathbf{d}^{\mathbf{T}} = \{u_1 \quad u_2 \quad u_3\}$. How do we do this? By creating what are called **augmented** elemental stiffness matrices. These matrices are expanded to size NxN if there are N nodal displacements in the global displacement vector **d**. Then if element 'e' is connected to nodes 'I' and 'J', we simply drop the element stiffness matrix in the submatrix formed by (I,I) (I,J); (J,I) (J,J). So for our 2-element 3-node example, for element '1' connected to nodes '1' and '2', we drop the the elemental stiffness matrix in the (1,1), (1,2); (2,1), (2,2) slots as follows:

$$\mathbf{K}_{\mathbf{1}}^{(aug)} = \begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} & 0 \\ k_{21}^{(1)} & k_{22}^{(1)} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The rest of the elements are kept zero. Doing the same thing for element '2' we have:

$$\mathbf{K}_{\mathbf{2}}^{(aug)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & k_{11}^{(2)} & k_{12}^{(2)} \\ 0 & k_{21}^{(2)} & k_{22}^{(2)} \end{bmatrix}.$$

The augmented matrices are therefore easily created once we have the element stiffness matrices.

Check now that the strain energy stored in element '1' can be written as:

$$U_1 = \frac{1}{2}\mathbf{d}^{\mathbf{T}}\mathbf{K}_{\mathbf{1}}^{(aug)}\mathbf{d} = \frac{1}{2}\{u_1 \quad u_2 \quad u_3\}\begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} & 0 \\ k_{21}^{(1)} & k_{22}^{(1)} & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$= \frac{1}{2}\left\{ k_{11}^{(1)}u_1^2 + k_{12}^{(1)}u_1u_2 + k_{21}^{(1)}u_1u_2 + k_{22}^{(1)}u_2^2 \right\}$$

which agrees with what we had earlier. Therefore, in general, we can write the energy stored in any element in terms of the global displacement vector using the augmented element stiffness matrices:

$$U = \sum_{all\ elements} \frac{1}{2}\mathbf{d}^T\mathbf{K}_{\mathbf{e}}^{(aug)}\mathbf{d}$$

$$= \frac{1}{2}\mathbf{d}^T\left\{ \sum_{all\ elements}\mathbf{K}_{\mathbf{e}}^{(aug)} \right\}\mathbf{d}$$

$$= \frac{1}{2}\mathbf{d}^T\mathbf{K}\mathbf{d}$$

where

$$\mathbf{K} = \left\{ \sum_{all\ elements}\mathbf{K}_{\mathbf{e}}^{(aug)} \right\}$$

is called the **global stiffness matrix** of the system, and it is just the matrix sum of all the augmented element stiffnes matrices.

The potential energy functional can therefore be written as:

$$\boxed{\Pi = \frac{1}{2}\mathbf{d}^T\mathbf{K}\mathbf{d} - \mathbf{d}^T\mathbf{F}}$$

-Now we need to minimize the potential energy functional with respect to each of the nodal displacements u_1, u_2, u_3,...That is, we get the system of equations:

$$\frac{\partial \Pi}{\partial u_1} = 0; \frac{\partial \Pi}{\partial u_2} = 0; \frac{\partial \Pi}{\partial u_3} = 0...$$

which are N equations if there are N nodal displacements. This is symbolically written as:

$$\frac{\partial \Pi}{\partial \mathbf{d}} = 0.$$

The resulting system of equations turns out to be particularly simple, namely:

$$\boxed{\mathbf{Kd} = \mathbf{F}}$$

To see this, it is best to work this out for the case of a system of rods with just one element. In this case:

$$\Pi = \frac{1}{2}\mathbf{d}^T\mathbf{K}\mathbf{d} - \mathbf{d}^T\mathbf{F}$$

$$= \frac{1}{2}\{u_1 \quad u_2\}\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} - \{u_1 \quad u_2\}\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$= \frac{1}{2}\left\{k_{11}u_1^2 + k_{12}u_1u_2 + k_{21}u_2u_1 + k_{22}u_2^2\right\} - \left(F_1u_1 + F_2u_2\right)$$

Now minimizing the above leads to:

$$\frac{\partial \Pi}{\partial u_1} = 0 \qquad \frac{1}{2}\left(2k_{11}u_1 + k_{12}u_2 + k_{21}u_2\right) - F_1 = 0$$

$$\frac{\partial \Pi}{\partial u_2} = 0 \qquad \frac{1}{2}\left(k_{12}u_1 + k_{21}u_1 + 2k_{22}u_2\right) - F_2 = 0$$

Recognizing that $k_{12}=k_{21}$ (the element stiffness matrices (††) are symmetric, and so are the augmented matrices, and consequently in general, the global stiffness matrix is symmetric), we get the desired result:

$$\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

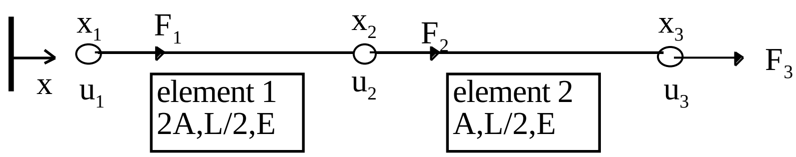
ie. $\mathbf{Kd}=\mathbf{F}$.

This works in general as well, and so we can formally stake a claim:

From the potential energy functional: $\Pi = \frac{1}{2}\mathbf{d}^T\mathbf{K}\mathbf{d} - \mathbf{d}^T\mathbf{F}$ setting $\frac{\partial \Pi}{\partial \mathbf{d}} = 0 \quad \mathbf{Kd} = \mathbf{F}$.

The beauty of this is that it works in more complex 2D and 3D settings as well. Before getting to that, however, let us see how we can use this type of formalism to rapidly (and eventually) automatically solve problems.

<u>Example:</u> Consider the two-element rod shown



The element stiffness matrices are:

$$K_1 = \begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} \\ k_{21}^{(1)} & k_{22}^{(1)} \end{bmatrix} = \frac{AE}{L}\begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix} \text{ and } K_2 = \begin{bmatrix} k_{11}^{(2)} & k_{12}^{(2)} \\ k_{21}^{(2)} & k_{22}^{(2)} \end{bmatrix} = \frac{AE}{L}\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix},$$

and so the global stiffness matrix is: $\mathbf{K} = \frac{AE}{L}\begin{bmatrix} 4 & -4 & 0 \\ -4 & 6 & -2 \\ 0 & -2 & 2 \end{bmatrix}$

Thus, we have: $\frac{AE}{L}\begin{bmatrix} 4 & -4 & 0 \\ -4 & 6 & -2 \\ 0 & -2 & 2 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$

These are called the *unconstrained* system equations. If you try to solve the above for the nodal displacements you will find that you cannot! The reason is we have not yet stipulated the boundary conditions for the problem! Let us do so now. Let node 1 be pinned, and let the only nodal force be acting on node 3 ie., F_3=P. If node '1' is pinned we have to have u_1=0, and so we can get rid of the equation corresponding to row '1':

$$\frac{AE}{L}\begin{bmatrix} 6 & -2 \\ -2 & 2 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ P \end{Bmatrix}$$

The above is the reduced equation for the *constrained* system . Note that it is exactly the same equation - as it had better be - that we had previously when we solved this problem in a more straight forward manner using the two-part Rayleigh-Ritz function.

It turns out that for computer implementation, it is simpler to get the constrained system equations by not reducing the size of the matrices as we did above, but rather by doing the following:

If node 'i' is constrained not to displace,

zero out the i'th row and i'th column of the stiffness matrix

pencil in a '1' at the i'th diagonal position

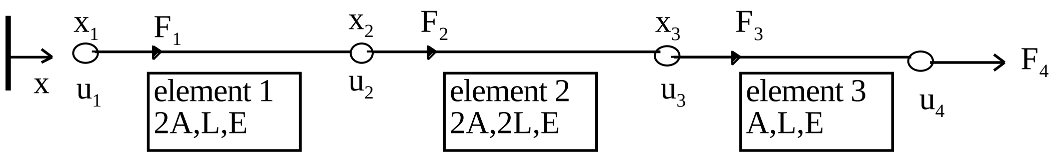
and zero out the i'th element of the load vector.

In the above, this yields: $$\frac{AE}{L}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 6 & -2 \\ 0 & -2 & 2 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ F_2 \\ F_3 \end{Bmatrix}$$

This is same constrained system equation as before (if F_2=0, and only F_3=P acts on the system), but it is not reduced in size. What we have done in effect is to replace the first equation with the condition u_1=0 (just expand out the first row to see this), which is just what we require for constraining node '1' in any case! Note that the rest of the above yields the same required system of equations for the unknown nodal displacements u_2 and u_3 as before.

The MATLAB code fem1d.m (available on the web) was used to solve this problem. You can do it by hand first, and then try the code.

<u>Exercise</u>: For the system shown:



(i) Get the global stiffness matrix for the unconstrained system

Ans: $\mathbf{K} = \frac{AE}{L}\begin{bmatrix} 2 & -2 & 0 & 0 \\ -2 & 3 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$

(ii) If nodes 1 and 4 are pinned, and a load of P acts on node '3', get the constrained system equations:

Ans: $\frac{AE}{L}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & -1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ P \\ 0 \end{Bmatrix}$

(iii) For A=0.1m^2, E=2.1x10^{11}N/m^2 and L=0.2m and P=100,000N, determine the nodal displacements and the stresses in the various elements of the rod.
{Hint: to get the stresses, note that once we have the nodal displacements, we can use $\varepsilon_e=\mathbf{B}_e\mathbf{d}_e$ to get the strain in each element, and then $\sigma_e=\mathbf{D}_e\varepsilon_e$ to get the stress.}

Ans: $\mathbf{d}^T$ = 1.0e-06 *[0 0.1905 0.5714 0] in meters