Problem Set 3

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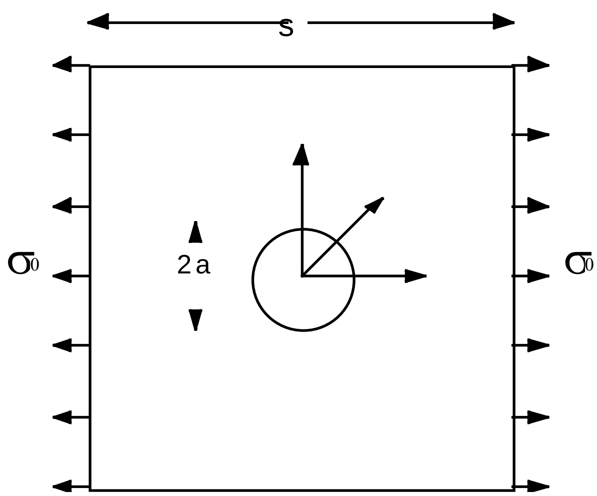
1. Stress Concentration: Consider a large, thin, square plate of sides s containing a small circular hole of radius a at its center. The plate is subjected to simple tensile stress of σ_o on its vertical edges. Locate a Cartesian (x-y) coordinate system as well as a polar r-θ coordinate system at the center of the hole as shown in the figure below.



(1.i) [2 points] Write down the boundary conditions on the square edges of the plate in terms of the Cartesian coordinate system. What are the boundary conditions in terms of the polar coordinate system on the circular edges of the hole?

(1.ii)[2 points] If the sides s of the plate are very large compared to the radius a of the hole, it is found that the stress distribution in the plate can be expressed in polar coordinates as:

$$\sigma_{rr} = \frac{\sigma_o}{2}\left[\left\{1-\frac{a^2}{r^2}\right\} + \left\{1 + \frac{3a^4}{r^4} - \frac{4a^2}{r^2}\right\}\cos 2\theta\right]$$
$$\sigma_{\theta\theta} = \frac{\sigma_o}{2}\left[\left\{1+\frac{a^2}{r^2}\right\} - \left\{1 + \frac{3a^4}{r^4}\right\}\cos 2\theta\right]$$
$$\sigma_{r\theta} = -\frac{\sigma_o}{2}\left[\left\{1 - \frac{3a^4}{r^4} + \frac{2a^2}{r^2}\right\}\sin 2\theta\right]$$

Verify that these satisfy the boundary conditions on the circular hole.

(1.iii) [2 points] On the edges of the hole, where r=a, plot (using MATLAB or similar package) the variation of normalized stress $\sigma_{\theta\theta} / \sigma_o$ as a function of θ. What are the maximum and minimum values of $\sigma_{\theta\theta} / \sigma_o$ and at what points along the hole do these occur?

(1.iv) [2 points] On the $\theta = 0$ line plot the normalized stresses σ_{rr} / σ_o , $\sigma_{\theta\theta} / \sigma_o$ and $\sigma_{r\theta} / \sigma_o$ as a function of normalized radial distance r/a in the range 1 to 15. What value does σ_{rr} tend to for large radial distances from the hole?

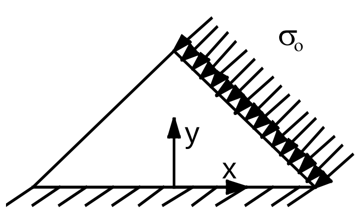
(1.v) [2 points] For large radial distances from the hole, r is much greater than hole radius a. Hence a/r is much less than one. Make this approximation in the equation for stresses given above and obtain expressions for the stresses (still in polar coordinates) that are valid far away from the hole.

(1.vi) [2 points] The stress transformation equations to go from polar to Cartesian coordinates are:

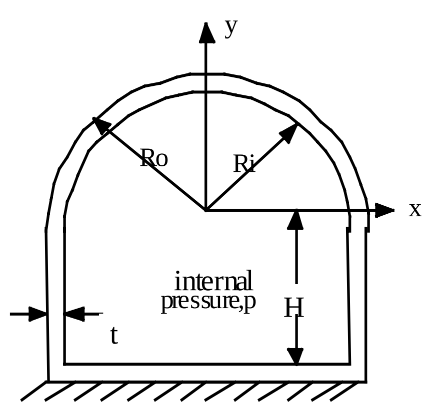
$$\sigma_{xx} = \frac{\sigma_{rr} + \sigma_{\theta\theta}}{2} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{2} \cos 2\theta - \sigma_{r\theta} \sin 2\theta$$
$$\sigma_{yy} = \frac{\sigma_{rr} + \sigma_{\theta\theta}}{2} - \frac{\sigma_{rr} - \sigma_{\theta\theta}}{2} \cos 2\theta + \sigma_{r\theta} \sin 2\theta$$
$$\sigma_{xy} = \frac{\sigma_{rr} - \sigma_{\theta\theta}}{2} \sin 2\theta + \sigma_{r\theta} \cos 2\theta$$

[Aside: Convince yourself that these are correct.]
Using the above transformation relations, express the stresses in the plate far away from the hole in Cartesian (x-y) coordinates. Verify that they do indeed satisfy the boundary conditions on the square edges of the plate.

2. Boundary Conditions: Specify the boundary conditions for each of the following cases in terms of the chosen xy-frame. Assume plane strain conditions prevail. [3+4 points]

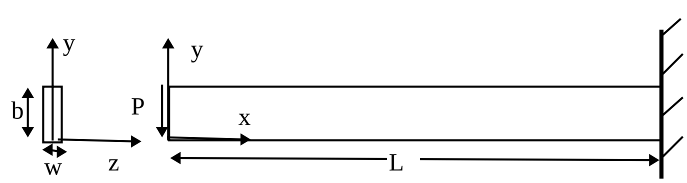


(2.i)



(2.ii)

3. Plane Stress: A cantilever beam of length L, width w and height b is fixed to a wall at x=L, and carries a shear load of magnitude P at the end x=0. The problem is treated as one of plane



stress since (w<<b). It is suggested that the Airy stress function for this problem is:
$\phi(x,y) = -\frac{P}{wb^3}x(3by^2 - 2y^3)$.
(3.i) [2 points] State the boundary conditions for this problem
(3.ii) [2 points] Obtain the stresses from the stress function, and check to see if the boundary conditions on the top and bottom of the beam (y=b and y=0) are satisfied.
(3.iii) [2 points] Show that the boundary condition on the loaded face (x=0) is satisfied only in an average sense.
Aside: The boundary condition at the fixed end x=L will not be satisfied exactly. So the solution we have is only approximate.

4. Biaxial Stress:
[5 points]A thin rectangular aluminum plate of sides a=0.3m, b=0.3m and thickness t=0.02m is loaded bi-axially as shown in the figure. Aluminum can be modeled as a linear elastic isotropic material with Young's modulus E=70GPa and Poisson's ration ν=0.33.
(4.i) What is the change in volume of the plate due to the loading?
(4.ii) Determine the plane along which the shear stress takes its largest value.