Problem Set 2

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1. Stress, Strain and Displacements:
[5 points] The stresses in an isotropic elastic body are given by:
$\sigma_{xx} = ay^2;\ \sigma_{yy} = -ax^2$ with all the other stress components being zero. Check that this satisfies the equations of equilibrium. What are the associated strains and displacements? How does a unit square in the xy-plane deform?

2. Conservation of Mass: A body is subject to the following deformation:

$$u_x = c_1 x,\ \ u_y = c_2 y,\ u_z = c_3 z, \qquad (*)$$

where the c's are small constants.

Consider a rectangular prism element of sides s_1, s_2, s_3 in the body (see Figure 2.2).

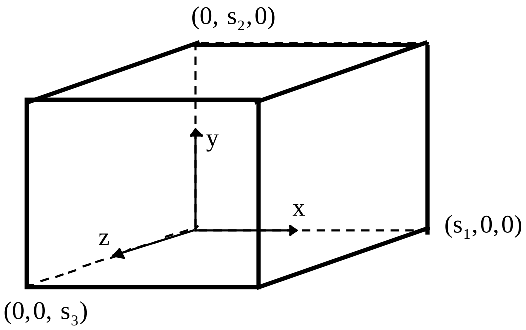

Figure 2.2

(2.i) [2 points] What is the volume of the prism element upon deformation?
(Simplification: $c_1 \ll 1$, $c_2 \ll 1$, $c_3 \ll 1$)

(2.ii) [2 points] Let us postulate that the mass of the prism must be conserved. (This is a law of nature analogous to Newton's laws, and consequently cannot be proved, but is assumed from experience to be true.) If the density of the material of the prism was ρ *before* deformation and ρ' upon deformation, show that these are related through:

$$\rho = \rho' \left(1 + \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right)$$

Remarks: This relation can be shown to hold for any arbitrary small deformation of a body, not just for the deformation given in (*).

(2.iii) [3 points] If $c_1=c_2=c_3=c$, constant, and the material is isotropic linear elastic, what are the stresses in the body?

3. Uniaxial Tension: A body undergoes a deformation such that the displacements through the body are given by:

$$u_x = \varepsilon\, x, \quad u_y = -\upsilon\varepsilon\, y, \quad u_x = -\upsilon\varepsilon\, z$$

where ε is a small constant and ν is the Poisson's ratio of the material.

(3.i) [1 point] How does a unit cube in the body deform in this case?

(3.ii) [1 point] What are the strains in the body?

(3.iii) [1 point] Assuming that the body is linear elastic, isotropic and homogenous, what are the stresses in the body?

(3.iv) [1 point] Is the body under static equilibrium with zero body forces?

Remarks: Can you see that this is the complete linear elasticity solution (ie satisfies all the equations and boundary conditions) to the problem of uniform stretching of a rod? Pictorially can you see how the rod is constrained, and how it is loaded?

4. Strain Transformation: An electrical strain gage is a device that measures normal strains (measure of change in length). It is essentially just a piece of conductor that is bonded to a body, and it stretches or compresses along with the body. As the length of the conductor changes, so does its resistance, and this can be measured using appropriate electrical circuitry.

Three strain gages are mounted on a body at point O along the OA, OB and OC directions as shown in Figure 2.4. The body is then subjected to a plane deformation, whereupon the gages read the following normal strains:
along direction OA, $\varepsilon_{OA} = 380\text{x}10^{-6}$,
along OB, $\varepsilon_{OB} = 400\text{x}10^{-6}$, and
along OC, $\varepsilon_{OC} = -600\text{x}10^{-6}$.

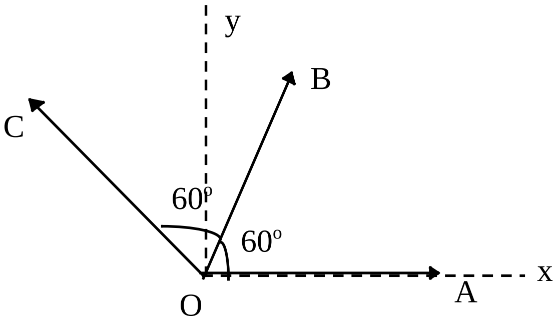

Figure 2.4

(4.i) [3 points] What is the state of strain at point O with respect to the xy frame?

(4.ii) [7 points] Along what line at point O is the normal strain largest, and what is its value? {*Hint*: You did something similar but for stresses in Pset 1.}

Remarks: Such an angular arrangement of 3 strain gages is called a strain gauge *rosette* , and these are very useful in measuring the complete (two dimensional) state of strain at a point in a body.

5. Cylindrical Polar Coordinates: For certain two-dimensional problems, it is convenient to use cylindrical polar coordinates. In order to characterize the geometry of deformation, we now keep track of *displacements* of points along the radial r and tangential θ directions as shown in Fig. 2.5 (in two dimensions only). Thus we have the displacement functions: $u_r(r,\theta)$, $u_\theta(r,\theta)$ that keep track of particle motion.

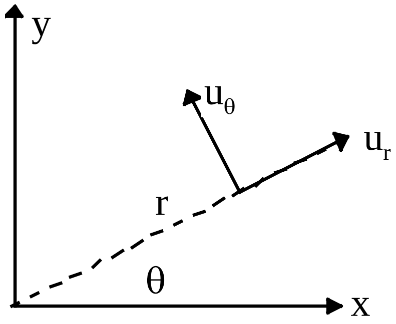

Fig. 2.5.1

By considering line elements that are along the radial and tangential directions before deformation, and monitoring their new position upon deformation, we can define strain components in polar coordinates:

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}$$

$$\varepsilon_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}$$

$$\varepsilon_{r\theta} = \varepsilon_{\theta r} = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r}\right)$$

(5.i) [2 points] What are these quantities a measure of? Express in words.

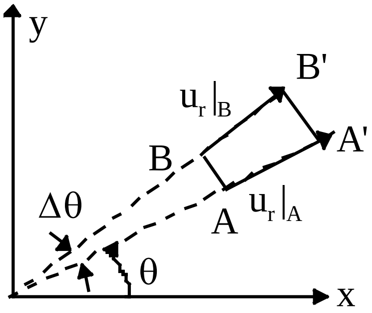

Fig. 2.5.2

(5.ii) [2 points] A body undergoes purely radial deformation (ie. every point moves radially out or in, and so u_r exists but $u_\theta=0$ everywhere as shown in the Fig. 2.6.2). What is the tangential normal strain $\varepsilon_{\theta\theta}$? Explain why a purely radial displacement field gives rise to a non-zero tangential strain.