Problem Set 1

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1. Stress Transformation Relations: Cauchy's theorem states that knowing the 9 stress components at a point with respect to one Cartesian reference frame (which defines three mutually perpendicular planes) we can obtain the stress components with respect to any other reference frame. To see how this works in general is algebraically a bit messy, so let us restrict attention to a body in a state of plane stress under static equilibrium with no applied body forces.

Suppose we know the stress components at any point P (see Figure 1.1) with respect to the xy-frame, i.e., we know σ_{xx}, σ_{yy}, $\sigma_{xy} = \sigma_{yx}$ at point P. We want to determine the intensity of internal force vector (more usually called the **stress vector** or **traction**) at point P across an arbitrary plane whose normal is rotated by an angle θ with respect to the x-axis. Equivalently, we want to determine the stress components at point P with respect to an x'y'-frame that is rotated with respect to the xy-frame.

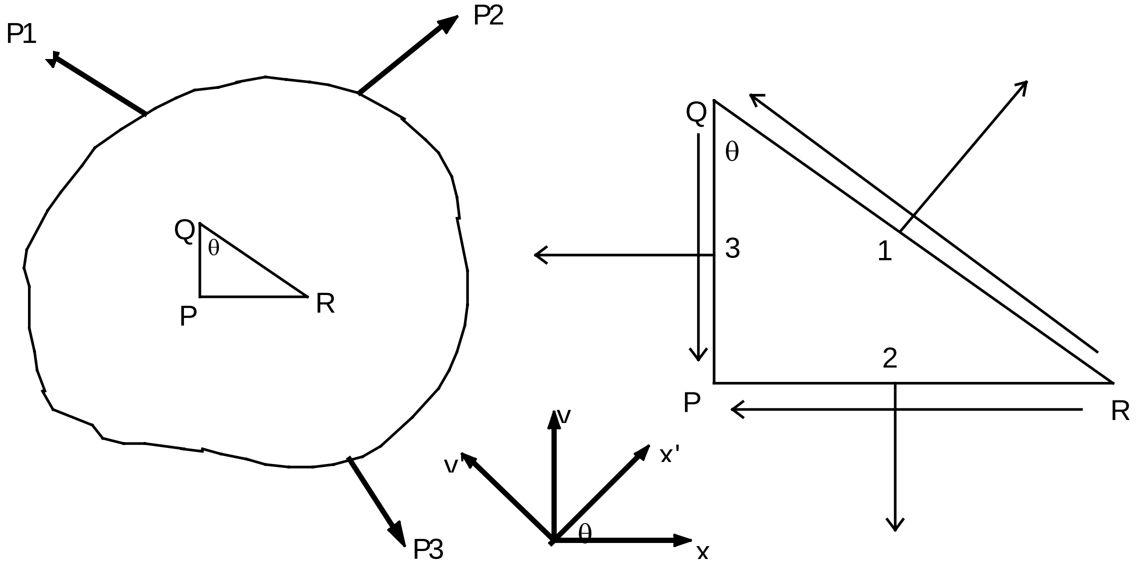


Figure 1.1

(1.i) [0 points] Consider the triangular element PQR of sides PQ = Δy, PR = Δx, QR = Δl. Let its thickness be unity. Note that $\Delta x = \Delta l \sin\theta$ and $\Delta y = \Delta l \cos\theta$.

(1.ii) [1 points] Draw the stress components acting on the various faces of the element PQR. Approximate the stresses on each face of the triangle by their values at the center point. That is, on face PR the stress components can be approximated by their values at the center point 2, and will be $(\sigma_{yy})_2$ and $(\sigma_{yx})_2$ etc. Note that it is most convenient to represent the stress components on inclined face QR in terms of the x'y'-frame. This is because the outward normal to QR is along the x'-direction.

(1.iii) [1 points] For the element PQR to be under static equilibrium, Newton's laws must hold. By considering force equilibrium about the x-direction, and using our standard procedure (of dividing out a length Δl and shrinking the element to zero *keeping its triangular shape*), show that we get:

$$\sigma_{x'x'} \cos\theta - \sigma_{x'y'} \sin\theta = \sigma_{xx} \cos\theta + \sigma_{yx} \sin\theta \quad (*)$$

where the location tags have been dropped since all points 1,2 and 3 coalesce to the same point P.

(1.iv) [0 points] By repeating the above exercise for force equilibrium along the y-direction you will get:

$$\sigma_{x'x'} \sin\theta + \sigma_{x'y'} \cos\theta = \sigma_{yy} \sin\theta + \sigma_{xy} \cos\theta \qquad (**)$$

(1.v) [1 point] By solving the algebraic equations (*) and (**), show that you get

$$\boxed{\sigma_{x'x'} = \sigma_{xx} \cos^2\theta + \sigma_{yy} \sin^2\theta + 2\,\sigma_{xy} \sin\theta \cos\theta}$$
$$\boxed{\sigma_{x'y'} = \sigma_{xy} \left(\cos^2\theta - \sin^2\theta\right) + (\sigma_{yy} - \sigma_{xx}) \sin\theta \cos\theta}$$

(1.vi) [0 points] A similar procedure can be adopted on another triangular element (which is the mirror image of the one we considered) to get:

$$\boxed{\sigma_{y'y'} = \sigma_{xx} \sin^2\theta + \sigma_{yy} \cos^2\theta - 2\,\sigma_{xy} \sin\theta \cos\theta}$$
$$\sigma_{y'x'} = \sigma_{x'y'}$$

These are the **stress transformation** equations in two dimensions.

(1.vii) In three-dimensions the stress transformation relations will look more complicated of course. The adventurous among you might want to try developing the stress transformation relations for the general three-dimensional case. You will have to consider a tetrahedral element in this case. In any case, please read the web notes on tensor transformation in three dimensions right now.

[2 points] The state of stress at a point with respect to an xyz Cartesian coordinate system is known to be:

$$\boldsymbol{\sigma} = \begin{bmatrix} 20 & 12 & -15 \\ 12 & 0 & 10 \\ -15 & 10 & 6 \end{bmatrix} \text{MPa}$$

Determine the stress components at the same point with respect to an x’y’z’ Cartesian coordinate system that is rotated 60^o clockwise about the z-axis.

Remarks, Asides etc:

(a) The stress transformation relations actually hold even if a body is subject to body forces and is under dynamic equilibrium. Can you see why?

(b) What we have just developed is a non-rigorous ``proof" of Cauchy's theorem. What are the fundamental principles or laws that have gone into your derivation? That is, on what principles does Cauchy's theorem rest?

(c) Using trigonometric formulas, express the two-dimensional stress transformation relations above in terms of 2θ. The resulting expressions are simpler to use in some cases, and anyway it is nice to have them for your records.

2. Principal Stresses: For a certain class of brittle materials, a simplistic failure criterion might be that if the normal stress at any point in the body *across any plane* reaches some critical value (depending on the material), the body will fracture. In this case, we will have to investigate the largest normal stresses at each point in the body after we have somehow obtained (through, say, a computer analysis of the structure) the stress components throughout the body with respect to some chosen coordinate frame.

To obtain the largest normal stresses, we can use the first of the stress transformation equations, and maximize with respect to orientation θ. It is convenient to first cast the stress transformation expression in terms of 2θ using trigonometric identities:

$$\sigma_{x'x'} = \frac{\sigma_{xx}+\sigma_{yy}}{2} + \frac{\sigma_{xx}-\sigma_{yy}}{2}\cos 2\theta + \sigma_{xy}\sin 2\theta$$

(2.i) [2 points] For what angle $\theta = \theta_p$ is $\sigma_{x'x'}$ a maximum or a minimum?
{Hint: The transformation relation above provides $\sigma_{x'x'}$ as a function of orientation θ. From elementary calculus, the maximum or minimum orientation occurs when:
$\left.\frac{d\sigma_{x'x'}}{d\theta}\right|_{\theta=\theta_p} = 0$.}

Remarks: The planes across which the normal stresses are maximum and minimum are called **principal planes** for the stress at that point, and the associated stresses are called **principal stresses**. The x'y' coordinate frame which corresponds to the principal planes is called the **principal frame** for the stress at that point.

(2.ii) [2 points] Show that the stresses in the principal frame are given by:

the maximum principal stress $$\boxed{\sigma_{x'x'}\big|_{\theta=\theta_p} \equiv \sigma_1 = \frac{\sigma_{xx}+\sigma_{yy}}{2} + \sqrt{\left[\frac{\sigma_{xx}-\sigma_{yy}}{2}\right]^2 + \sigma_{xy}^2}}$$

the minimum principal stress $$\boxed{\sigma_{y'y'}\big|_{\theta=\theta_p} \equiv \sigma_2 = \frac{\sigma_{xx}+\sigma_{yy}}{2} - \sqrt{\left[\frac{\sigma_{xx}-\sigma_{yy}}{2}\right]^2 + \sigma_{xy}^2}}$$

and the shear stress $\boxed{\sigma_{x'y'}\big|_{\theta=\theta_p} = 0}$.

Remarks: Note that the shear stress is zero across the principal planes.

(2.iii) [2 points] The state of stress at a point in a thin plate is given to be:
$\sigma_{xx} = 50MPa; \quad \sigma_{yy} = -25MPa; \quad \sigma_{xy} = 30MPa;$
Determine the maximum and minimum principal stresses and pictorially show the planes across which these act.

(2.iv) [2+2 points] If the state of stress is three-dimensional, the problem of determining principal stresses is a bit more involved. Without proof, let me just state that **the principal stresses are given by the *eigenvalues* of the stress tensor and the principal planes (defined by their normal vectors) are given by its *eigenvectors*.** For the state of stress given in (2.iii) above:

$$\sigma = \begin{bmatrix} 50 & 30 & 0 \\ 30 & -25 & 0 \\ 0 & 0 & 0 \end{bmatrix} \textbf{ MPa}$$

determine the eigenvalues and eigenvectors and check if the results agree with what you obtained in (2.iii). Also determine the principal stresses and planes at a point where the stress tensor is fully three-dimensional as follows:

$$\sigma = \begin{bmatrix} 40 & 24 & -30 \\ 24 & 0 & 20 \\ -30 & 20 & 12 \end{bmatrix} \text{MPa}$$

3. Equilibrium: Determine whether the following stress distributions can exist for a body in static equilibrium :
(3.i) [2 points] All the c's are constants.
$\sigma_{xx} = -2\, c_1\, x\, y;\ \ \sigma_{yy} = c_2\, z^2;\ \ \sigma_{xy} = c_1\,(c_2 - y^2) + c_3\, x\, z;\ \ \sigma_{zz} = 0;\ \ \sigma_{xz} = 0;\ \sigma_{yz}=0;$ (no body forces)

(3.ii) [3 points] Determine the body force density distribution required such that the following state of stress represents static equilibrium of a body:
$\sigma_{xx} = x^2 + 2y;\ \ \sigma_{yy} = xy-y^2z;\ \ \sigma_{xy} = -xy^2+1;\ \ \sigma_{zz} = x^2-z^2;\ \ \sigma_{xz} = xz-2x^2y;\ \sigma_{yz}=0.$

4. Airy stress function: A body is in a state of plane stress, and the stress components are given in terms of a function Φ (called the Airy stress function) such that:

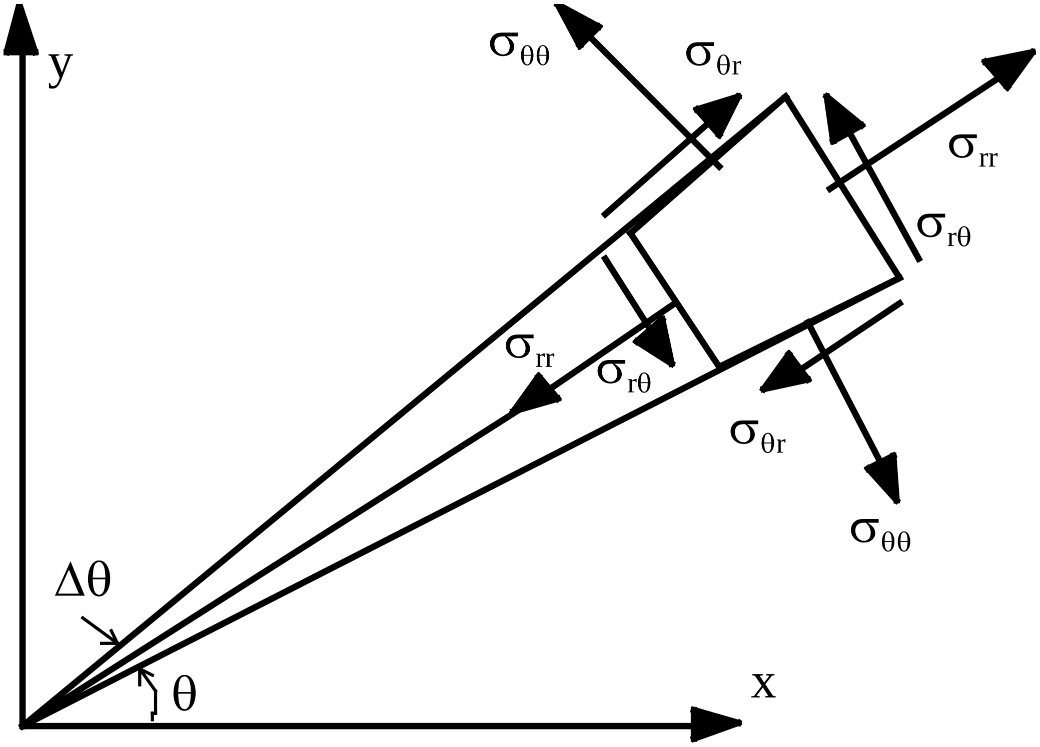
$$\sigma_{xx} = \frac{\partial^2 \Phi}{\partial y^2}, \qquad \sigma_{yy} = \frac{\partial^2 \Phi}{\partial x^2}, \qquad \sigma_{xy} = -\frac{\partial^2 \Phi}{\partial x \partial y}$$

(4.i) [2 points] Show that the stress components derived as above from the Airy stress function are in static equilibrium with no body forces.
(4.ii) [3 points] Obtain the stress components if:

$$\Phi = -\frac{F}{d^3}\, x\, \{\, 3\, d\, y^2 - 2\, y^3 \,\}$$

where F and d are constants.

5. Cylindrical Polar Coordinates: For some two-dimensional problems involving curved boundaries, it is advantageous to use cylindrical polar coordinates. Consider the wedge element centered at (r, θ), and bounded in the radial (r) directions and the tangential (θ) directions as shown in the figure.



(5.i) [0 points] By considering the static equilibrium of the element under zero body forces, we find that

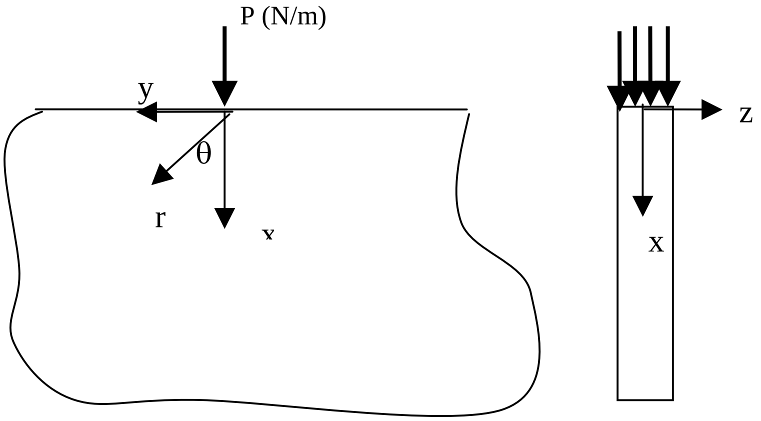
$$\sigma_{r\theta} = \sigma_{\theta r}$$

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0$$

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{2\,\sigma_{r\theta}}{r} = 0$$

[*Aside*: Convince yourself that this makes sense. For the rest of this course, I am going to assume that you are convinced!]

(5.ii) [3 points] A thin, very large plate carries a line load P (load per unit thickness of plate). The plate can be considered to be under plane stress conditions.



With respect to the polar coordinate system shown, it is found that the stresses in the plate are:

$$\sigma_{rr} = -\frac{2P}{\pi}\frac{\cos\theta}{r}, \qquad \sigma_{r\theta} = 0, \qquad \sigma_{\theta\theta} = 0.$$

Do these stresses satisfy static equilibrium under zero body forces?

Yet another aside: A real life situation where this problem might be applicable is that of a train wheel (providing the line load) resting on a rail.
Q?: What are the stresses at the points of application of the loads (at r = 0)? What gives?