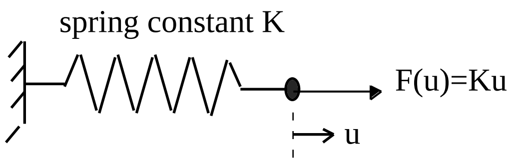
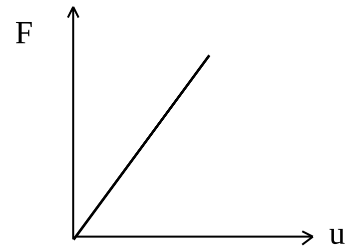
ENERGY PRINCIPLES

6.1 STRAIN ENERGY:

Consider a linear spring :





This is a spring whose stretch 'u' and load F are proportional through a spring constant K (N/m).

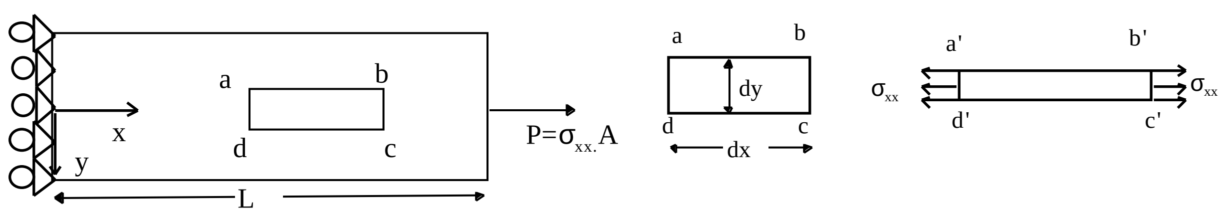
The load F is said to be applied *statically* if the spring is in static equilibrium at all times as the load is increased from an initial value of 0 to its final value of F. The work done by the applied load in moving through a distance 'u' in a statical loading process is therefore:

$$W = \int_0^u F(u)\mathrm{d}u = \frac{1}{2}Ku^2 = \frac{1}{2}Fu\,.$$

For a linear spring that is non-dissipative (elastic), this work done is stored as **extensional energy** in the spring which can be recovered on unloading.

Claim: A linear elastic solid can be thought of as a continuum of linear elastic "springs" As the body deforms on application of external loads, the work done by these loads is stored in the body as **strain energy** which can be recovered upon unloading.

Reason why: To see this, consider a cylindrical body of length L and cross-sectional area A that is in uniaxial tension.The loads are applied statically such that in going from the unloaded state to its final state where the stress is σ_{xx} and the strain is: $\varepsilon_{xx} = \sigma_{xx} / E$, the body is always in static equilibrium.



Consider an infinitesimal "spring element" abcd of size dx dy dz shown. The force acting on the spring is $\{\sigma_{xx}\ dy\ dz\}$. Its final extension is ε_{xx} dx. In computing the work done by the force in stretching this elemental spring to its final value,we consider the final strain ε_{xx} to have been reached through a series of strain increments $d\varepsilon_{xx}$.

Thus $\Delta W = \int_0^{\varepsilon_{xx}} \{\sigma_{xx} dy.dz\}\{d\varepsilon_{xx}.dx\} = \Delta U$ which is stored as strain energy, and the Δ's remind us that this is just for one infinitesimal spring element. Summing up the energy stored in all the spring elements that make up the body, the total strain energy stored in uniaxial extension is:

$$U = \int_V \int_0^{\varepsilon_{xx}} \{\sigma_{xx}.d\varepsilon_{xx}\} \underbrace{dx.dy.dz}_{dV}$$

We define a **strain energy density** which is strain energy per unit volume as:

$$U_o = \int_0^{\varepsilon_{xx}} \{\sigma_{xx}.d\varepsilon_{xx}\} ,$$

and therefore the total strain energy is just:

$$U = \int_V U_o dV$$

For a linear elastic solid, using Hooke's law, we find that the strain energy becomes:

$$U_o = \int_0^{\varepsilon_{xx}} \{\sigma_{xx}.d\varepsilon_{xx}\} = \int_0^{\varepsilon_{xx}} \{E\varepsilon_{xx}.d\varepsilon_{xx}\} = \frac{1}{2} E\varepsilon_{xx}^2 = \frac{1}{2}\sigma_{xx}\varepsilon_{xx}$$

Generalization to three-dimensional states of stress and strain yields an expanded version for the strain energy density.

$$U_o = \frac{1}{2}\{\sigma_{xx}\varepsilon_{xx} + \sigma_{yy}\varepsilon_{yy} + \sigma_{zz}\varepsilon_{zz} + 2\sigma_{xy}\varepsilon_{xy} + 2\sigma_{yz}\varepsilon_{yz} + 2\sigma_{zx}\varepsilon_{zx}\}$$

The first three terms involving normal stresses can be thought of as arising from linear spring elements (the ones that stretch) along the x-, y- and z-directions respectively, and the last three terms involving shear stresses arise from "torsional" spring elements (the ones that twist). Can you guess why we have the factor '2' associated with the shear terms?

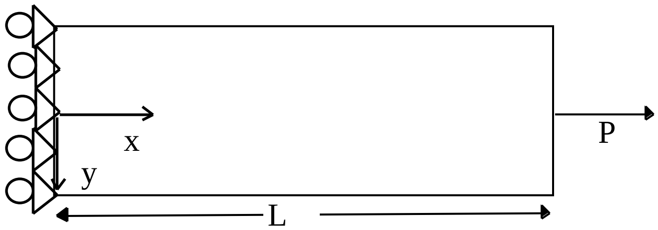
The strain energy density for a general state of stress and strain can be written more compactly in matrix notation:

$$\boxed{U_o = \frac{1}{2}\varepsilon^T\sigma = \frac{1}{2}\varepsilon^T\mathbf{D}\varepsilon}.$$

where $\boldsymbol{\varepsilon}^{\mathbf{T}} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{yy} & \varepsilon_{zz} & \gamma_{xy} & \gamma_{yz} & \gamma_{zx} \end{bmatrix}$, and $\boldsymbol{\sigma}^{\mathbf{T}} = \begin{bmatrix} \sigma_{xx} & \sigma_{yy} & \sigma_{zz} & \sigma_{xy} & \sigma_{yz} & \sigma_{zx} \end{bmatrix}$, and **D** is the elasticity matrix. (Note use of the engineering shear strains.)

Therefore the total strain energy stored in the body is: $\boxed{U = \frac{1}{2}\int_V \boldsymbol{\varepsilon}^{\mathrm{T}} \mathbf{D} \boldsymbol{\varepsilon} \; dV}$.

6.2 STRAIN ENERGY IN UNIAXIAL STRETCHING OF A ROD:



Consider a rod of initial length L and cross-sectional area A(x) - not necessarily of uniform cross-section- loaded by a load P. If the variation in the cross-section is not too abrupt, we can say that the only important quantities are $\sigma_{xx} = E\varepsilon_{xx} = E\dfrac{du_x}{dx}$.

The strain energy density is therefore: $U_o = \dfrac{1}{2}\sigma_{xx}\varepsilon_{xx} = \dfrac{1}{2}E\left\{\dfrac{du_x}{dx}\right\}^2$ and the total strain energy in the rod is: $U = \int\limits_V U_o.dV = \int\limits_V \dfrac{1}{2}E\left\{\dfrac{du_x}{dx}\right\}^2.dV = \int\limits_{x=0}^{x=L}\dfrac{EA(x)}{2}\left\{\dfrac{du_x(x)}{dx}\right\}^2.dx$

6.3 STRAIN ENERGY IN BENDING OF A BEAM:

According to the Bernoulli-Euler beam theory, the only important components are σ_{xx} and ε_{xx}. The strain energy density is therefore:

$$U_o = \frac{1}{2}\sigma_{xx}\varepsilon_{xx} = \frac{1}{2}\left\{-Ey\frac{d^2u_y(x)}{dx^2}\right\}\left\{-y\frac{d^2u_y(x)}{dx^2}\right\} = \frac{E}{2}y^2\left\{\frac{d^2u_y(x)}{dx^2}\right\}^2$$

and so the total strain energy stored in the beam is:

$$U = \int\limits_V \frac{E}{2}y^2\left\{\frac{d^2u_y(x)}{dx^2}\right\}^2 dV$$

$$= \int\limits_V \frac{E}{2}y^2\left\{\frac{d^2u_y(x)}{dx^2}\right\}^2 dx.\underbrace{dydz}_{dA}$$

$$= \int\limits_{x=0}^{x=L}\frac{E}{2}\left\{\frac{d^2u_y(x)}{dx^2}\right\}^2\underbrace{\left\{\int\limits_A y^2dA\right\}}_{I_z} dx$$

$$= \int\limits_{x=0}^{x=L}\frac{EI_z}{2}\left\{\frac{d^2u_y(x)}{dx^2}\right\}^2 dx$$

The nice thing about the above expression is that we can actually allow the cross-section of the beam to vary, ie. $I_z(x)$, and the above expression will still hold true.

6.4 THE POTENTIAL ENERGY FUNCTIONAL:

We will now postulate a principle that will enable us to solve structural deformation problems in a way that is quite different from the force balance approach we have been taking so far. First, a few words about functions and functionals.

(i) Functions: A function y(x) eats independent variables x and spits out values for the dependent variable y.

Examples: y(x) = sin x, y(x) = e^{-ax} etc.

One can get the maxima and minima of these functions using the tools of ordinary calculus with which you are all familiar.

(ii) Functionals: A functional eats entire functions over some range of the independent variable, and spits out values.

Example:

$$\mathbf{I}\{y(x)\} = \int_0^1 \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2} dx$$

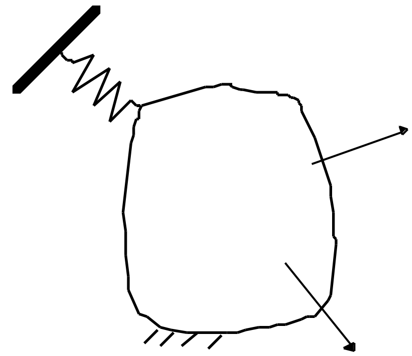
You can easily figure out that the above measures the length of a smooth curve y(x) between x=0 and x=1.

Question: What smooth function y(x) that passes through y = a when x=0 and y=b when x=1 minimizes the functional **I**{y(x)}?

Answer: The straight line y(x) = (b-a) x + a.

Why? How did you know this? If we try to rigorously show that we have the right answer, we will end up creating a new tool of analysis called the calculus of variations. This is a very powerful and extremely fascinating subject of mathematics that I urge you to delve into when you get the opportunity. Unfortunately, now is not the time for that adventure. However, just keep in mind that functionals process entire function or functions to produce a value. Now for one very important functional in our theory.

Consider the structural system shown consisting of a linear elastic body that deforms under the action of some applied loads and some constraints. We will assume that there are no energy dissipative elements in the system, but there may be other elastic (energy storing) things like linear springs etc.



The displacements, strains and stresses that develop in the body are all functions of the spatial coordinates of the body.

displacements (3): u_x, u_y, u_z

strains (6): $\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, \varepsilon_{xy}, \varepsilon_{yz}, \varepsilon_{zx}$

and stresses (6): $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{yz}, \sigma_{zx}$

Define the **potential energy functional:**

$$\boxed{\Pi = U - W}$$

Here U is the total stored energy in the system which includes the strain energy stored in the elastic structure plus the energy stored in any springs and other energy storing devices in the system.

The term $W = \sum_{\text{all loads}} \mathrm{P}_\mathrm{i}.e_i$ we will call the virtual work where P_i are the net applied loads and e_i are the net displacements under the corresponding loads in the direction of the loads. The virtual work is a work-like quantity, but note that it is not the actual work done by the applied loads in deforming the body statically. (Why not?). Π is a functional because it eats whole the displacement, stress and strain functions thrown to it in some combination, and produces a single number.

6.5 THE PRINCIPLE OF MINIMUM POTENTIAL ENERGY (PMPE):

Of all the possible displacement functions that are sufficiently smooth (for compatibility) and which satisfy the *geometric* boundary conditions of the system, the set that corresponds to the stable static equilibrium of the system is the one that minimizes the potential energy functional of the system.

Symbolically, we say: $\delta\Pi = \delta(U - W) = 0$ at equilibrium.

Remarks:

(i) Why is this PMPE true? For linear elastic systems, if we postulate Newton's laws of equilibrium, then we can use the tools of calculus of variations to prove that PMPE must be true. Conversely, if we postulate PMPE, then we can prove that Newton's laws of equilibrium must follow. Since we (you) do not have the tools needed to show this, we are forced to postulate PMPE as a law of nature. (Actually, we just trade away Newton's laws --which, if you recall, we postulated without proof-- in exchange for PMPE; we just need postulate one of them, not both.)

By means of several examples we will see that PMPE indeed agrees with results we have obtained previously using Newton's laws. Thus, while PMPE is not intuitive now, you will soon take it for granted the same way you have accepted Newton's laws.

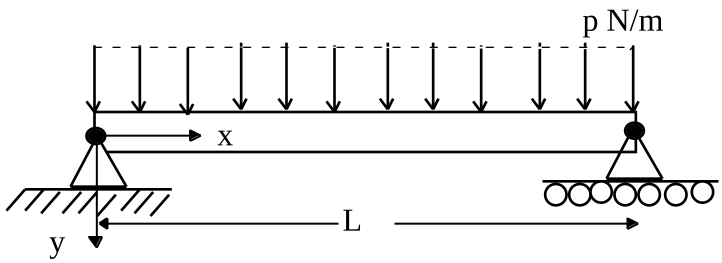
(ii) How is PMPE useful? Given a structural system, we create a bag full of all possible candidate displacement functions that are sufficiently smooth and which satisfy the geometric boundary conditions. We ignore load boundary conditions. Then using the strain-displacement and the stress-strain relations we obtain the strains and stresses. We can then evaluate the potential energy functional Π for the system. We do this for every candidate displacement in our bag, and get that particular one that minimizes Π. PMPE says that this is the actual solution to the problem.

(iii) In practice, we will restrict the bag of candidate displacement functions to something reasonable (such as sines, cosines, polynomials etc) and seek the Π minimizer from this limited bag. This is called the Rayleigh-Ritz technique, and it will yield the best possible approximation to the actual solution that is available from this limited bag. There are some clever ways of getting more accurate results, and we will see how these lead to the finite element method that you have probably heard about.

Now for some action using PMPE.

6.6 AN EXACT SOLUTION TO A BEAM PROBLEM USING PMPE:

To gain confidence in the PMPE method, let us consider the following beam bending problem. Let the beam be of uniform flexural rigidity EI_z.



(i) *Bag of displacements:* Seek a beam deflection function in the form of a Fourier sine series:

$$u_y(x) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L} \qquad (\dagger)$$

where the a_n's are undetermined parameters. This clearly satisfies the geometric boundary conditions (gbc) that $u_y(0) = 0;\ \ u_y(L) = 0$, and that is all that we care about as far as PMPE is concerned. We do not worry about load boundary conditions (such as the fact here that pins cannot support moments).

Thus (†) is a legal bag of candidate displacements which contains a whole bunch (an infinity of them actually) of functions as we change the a_n's.

(ii) *Calculate the Potential Energy Functional:*

The stored energy is:

$$U = \int_{x=0}^{L} \frac{EI_z}{2}\left\{\frac{d^2u_y}{dx^2}\right\}^2 dx$$

$$= \frac{EI_z}{2}\int_0^L \left\{\sum_{n=1}^{\infty} a_n\left(\frac{n\pi}{L}\right)^2(-1)\sin\frac{n\pi x}{L}\right\}^2 dx$$

$$= \frac{EI_z}{2}\int_0^L \left\{\sum_{n=1}^{\infty} a_n\left(\frac{n\pi}{L}\right)^2(-1)\sin\frac{n\pi x}{L}\right\}\left\{\sum_{m=1}^{\infty} a_m\left(\frac{m\pi}{L}\right)^2(-1)\sin\frac{m\pi x}{L}\right\} dx$$

$$= \frac{EI_z}{2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left\{a_n a_m\left(\frac{n\pi}{L}\right)^2\left(\frac{m\pi}{L}\right)^2\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx\right\}$$

$$= \frac{EI_z}{2}\sum_{n=1}^{\infty} a_n^2\left(\frac{n\pi}{L}\right)^4\frac{L}{2}$$

The virtual work term becomes:

$$
\begin{aligned}
W &= \int_{x=0}^{L} p.u_y(x)\ dx \\
&= \int_{x=0}^{L} p.\left\{ \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L} \right\} dx \\
&= p.\left\{ \sum_{n=1}^{\infty} a_n \int_{x=0}^{L} \sin\frac{n\pi x}{L} dx \right\} \\
&= \frac{pL}{\pi} \sum_{n=1}^{\infty} \frac{a_n}{n}(1-\cos n\pi)
\end{aligned}
$$

The potential energy functional is therefore:

$$\Pi = \frac{\pi^4 EI_z}{4L^3} \sum_{n=1}^{\infty} n^4 a_n^2 - \frac{pL}{\pi} \sum_{n=1}^{\infty} \frac{a_n}{n}(1-\cos n\pi)$$

(iii) *Minimize the potential energy functional:* This simply requires us to find that set of a_n's that makes Π a minimum. This is a calculus problem that we can handle. Setting:

$$\frac{\partial \Pi}{\partial a_n} = 0; \ \ n = 1,2,3... \qquad \frac{\pi^4 EI_z}{4L^3} n^4 (2\, a_n) - \frac{pL}{\pi}\frac{1}{n}(1-\cos n\pi) = 0; \qquad n = 1,2,3...$$

Solving which we get

$$a_n = \frac{2}{\pi^5}\frac{pL^4}{EI_z}\frac{1}{n^5}(1-\cos n\pi); \qquad n = 1,2,3...$$

Therefore the beam deflection is:

$$\boxed{u_y(x) = \frac{2}{\pi^5}\frac{pL^4}{EI_z} \sum_{n=1}^{\infty} \left\{ \frac{1}{n^5}(1-\cos n\pi)\sin\frac{n\pi x}{L} \right\}}$$

<u>Remarks</u>:

(i) Is this the correct solution? It looks nothing like the "exact" solution we would have got using the Bernoulli-Euler beam theory (you did this on your ownin Pset 4):

$$\boxed{u_y^{BE}(x) = \frac{pL^4}{24EI_z}\left\{ \left(\frac{x}{L}\right)^4 - 2\left(\frac{x}{L}\right)^3 + \left(\frac{x}{L}\right) \right\}}.$$

But in fact, the two solutions are identical! You can plot both and see that this is true. Or else, let us consider just the mid-point deflection:

$$u_y^{BE}(L/2) = \frac{5}{384}\frac{pL^4}{EI_z} = \frac{1}{76.8}\frac{pL^4}{EI_z}$$

and from the PMPE solution

$$u_y^{PMPE}(L/2) = \frac{4}{\pi^5}\frac{pL^4}{EI_z}\left\{1-\frac{1}{3^5}+\frac{1}{5^5}-\frac{1}{7^5}+\ldots\right\}$$

Compare this with:

$$u_y^{PMPE}(L/2) = \frac{4}{\pi^5}\frac{pL^4}{EI_z}\left\{1-\frac{1}{3^5}+\frac{1}{5^5}-\frac{1}{7^5}+\ldots\right\}$$

$$\approx \frac{1}{76.6}\frac{pL^4}{EI_z} \quad \text{using only one term in the series}$$

Not bad! Cranking the PMPE machinery provided us the same solution as using Newton's laws (which gave us the Bernoulli-Euler theory); so maybe we can start to trust PMPE. Of course, in this case, you might argue that Bernoulli-Euler would have been easier. But what if the beam had not been uniform (ie., EI_z varied with cross-section), or if the loading function p(x) were not a constant but something more involved?

(ii) Note also that in this case, only one-term provides us a very reasonable approximation to the Bernoulli-Euler solution. This gives us our next flash of inspiration. If we had started out with just one term, our life would have been a lot simpler in calculating the potential energy functional (we would have been rid of the series summation stuff)! In fact, this is what we do in a Rayleigh-Ritz analysis.

Rayleigh-Ritz Procedure:

(i) *Create a bag of candidate displacements*: Assume a solution containing undetermined Rayleigh-Ritz parameters a_n, and satisfying the geometric boundary conditions (gbc) of the problem. Ignore load boundary conditions.

(ii) *Compute the potential energy functional*: Π as function of the undetermined Rayleigh-Ritz parameters $\Pi(a_1, a_2, \ldots a_n)$.

(iii) *Minimize the potential energy functional* with respect to the Rayleigh-Ritz parameters a_n

$$\frac{\partial \Pi}{\partial a_1} = 0; \quad \frac{\partial \Pi}{\partial a_2} = 0; \quad \ldots \frac{\partial \Pi}{\partial a_n} = 0;$$

and solve the resulting system for the a_ns. Technically speaking, one must of course check whether this corresponds to the minimum, but we rarely bother doing this.