THE EQUATIONS OF LINEAR ELASTICITY IN CARTESIAN COORDINATES: Three-dimensional deformation

Displacements (3): $\mathbf{u}^{\mathbf{T}} = \begin{bmatrix} u_x & u_y & u_z \end{bmatrix}$

Strains (6): $\varepsilon^{\mathbf{T}} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{yy} & \varepsilon_{zz} & \gamma_{xy} & \gamma_{yz} & \gamma_{zx} \end{bmatrix}$

Stresses (6): $\sigma^{\mathbf{T}} = \begin{bmatrix} \sigma_{xx} & \sigma_{yy} & \sigma_{zz} & \sigma_{xy} & \sigma_{yz} & \sigma_{zx} \end{bmatrix}$

EQUILIBRIUM EQUATIONS (3):

$$\begin{aligned}
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + f_x &= \rho \ddot{u}_x \\
\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + f_y &= \rho \ddot{u}_y \\
\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + f_z &= \rho \ddot{u}_z
\end{aligned} \qquad (1)$$

where a superposed dot represents a time derivative, and ρ is material density.

LINEAR ELASTIC, ISOTROPIC MATERIAL RESPONSE (6):

$$\sigma = \mathbf{D}\varepsilon$$

$$\text{where } \mathbf{D} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix}
1 & \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 0 & 0 & 0 \\
\frac{\nu}{1-\nu} & 1 & \frac{\nu}{1-\nu} & 0 & 0 & 0 \\
\frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)}
\end{bmatrix} \qquad (2)$$

STRAIN-DISPLACEMENT RELATIONS (6):

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} \qquad \gamma_{xy} = \gamma_{yx} = \left\{ \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right\}$$

$$\varepsilon_{yy} = \frac{\partial u_y}{\partial y} \qquad \gamma_{yz} = \gamma_{zy} = \left\{ \frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z} \right\} \qquad (3)$$

$$\varepsilon_{zz} = \frac{\partial u_z}{\partial z} \qquad \gamma_{xz} = \gamma_{zx} = \left\{ \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \right\}$$

Note: In lieu of the strain-displacement relations, we may use the compatibility conditions for the strains. For any given problem, we must solve for the fifteen unknowns using the above set of fifteen equations subject to the appropriate boundary constraints (either applied forces or displacements).

THE EQUATIONS OF LINEAR ELASTICITY IN CARTESIAN COORDINATES: PLANE STRESS

Displacements: $\mathbf{u}^T = \left[u_x(x,y) \quad u_y(x,y)\right]$

Strains: $\varepsilon^T = \left[\varepsilon_{xx}(x,y) \quad \varepsilon_{yy}(x,y) \quad \gamma_{xy}(x,y)\right]$

Stresses: $\sigma^T = \left[\sigma_{xx}(x,y) \quad \sigma_{yy}(x,y) \quad \sigma_{xy}(x,y)\right]$

Equilibrium Equations: (static equilibrium, no body forces):

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0$$

Strain-Displacement Relations:

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}, \quad \varepsilon_{yy} = \frac{\partial u_y}{\partial y}, \quad \gamma_{xy} = \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)$$

Reduced Stress-Strain Relation:

$$\sigma = \mathbf{D}\varepsilon \quad \text{where} \quad \mathbf{D} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}$$

Auxiliary Equation: The through-thickness strain is given by

$$\varepsilon_{zz} = -\frac{\nu}{E}\left(\sigma_{xx} + \sigma_{yy}\right)$$

from which one can obtain the out-of-plane displacement u_z via integration: $u_z = \int \varepsilon_{zz}\, dz$.

The Airy Stress Function Φ(x,y): Defining an Airy stress function through:

$$\sigma_{xx} = \frac{\partial^2 \Phi}{\partial y^2}, \quad \sigma_{yy} = \frac{\partial^2 \Phi}{\partial x^2}, \quad \sigma_{xy} = -\frac{\partial^2 \Phi}{\partial x \partial y}$$

The governing differential equation for plane stress then becomes:

$$\nabla^4 \Phi \equiv \frac{\partial^4 \Phi}{\partial x^4} + \frac{\partial^4 \Phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \Phi}{\partial y^4} = 0 .$$

THE EQUATIONS OF LINEAR ELASTICITY IN CARTESIAN COORDINATES: PLANE STRAIN

Displacements: $\mathbf{u}^T = [u_x(x,y) \quad u_y(x,y)]$

Strains: $\varepsilon^T = [\varepsilon_{xx}(x,y) \quad \varepsilon_{yy}(x,y) \quad \gamma_{xy}(x,y)]$

Stresses: $\sigma^T = [\sigma_{xx}(x,y) \quad \sigma_{yy}(x,y) \quad \sigma_{xy}(x,y)]$

Equilibrium Equations: (static equilibrium, no body forces):

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0$$

Strain-Displacement Relations:

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}, \quad \varepsilon_{yy} = \frac{\partial u_y}{\partial y}, \quad \gamma_{xy} = \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right)$$

Reduced Stress-Strain Relation:

$$\sigma = \mathbf{D}\varepsilon \text{ where } \mathbf{D} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \nu/(1-\nu) & 0 \\ \nu/(1-\nu) & 1 & 0 \\ 0 & 0 & (1-2\nu)/2(1-\nu) \end{bmatrix}$$

Auxiliary Equation: The through-thickness normal stress needed to obtain a state of plane deformation is:

$$\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy})$$

The Airy Stress Function $\Phi(x,y)$: Defining an Airy stress function through:

$$\sigma_{xx} = \frac{\partial^2 \Phi}{\partial y^2}, \quad \sigma_{yy} = \frac{\partial^2 \Phi}{\partial x^2}, \quad \sigma_{xy} = -\frac{\partial^2 \Phi}{\partial x \partial y}$$

The governing differential equation for plane stress then becomes:

$$^4\Phi \equiv \frac{\partial^4 \Phi}{\partial x^4} + \frac{\partial^4 \Phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \Phi}{\partial y^4} = 0 .$$

THE EQUATIONS OF LINEAR ELASTICITY IN CYLINDRICAL POLAR COORDINATES: PLANE STRESS

Displacements: $\mathbf{u}^T = \left[u_r(r,\theta) \quad u_\theta(r,\theta)\right]$

Strains: $\varepsilon^T = \left[\varepsilon_{rr}(r,\theta) \quad \varepsilon_{\theta\theta}(r,\theta) \quad \gamma_{r\theta}(r,\theta)\right]$

Stresses: $\sigma^T = \left[\sigma_{rr}(r,\theta) \quad \sigma_{\theta\theta}(r,\theta) \quad \sigma_{r\theta}(r,\theta)\right]$

Equilibrium Equations: (static equilibrium, no body forces):

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0$$

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{2\sigma_{r\theta}}{r} = 0$$

Strain-Displacement Relations:

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}; \quad \varepsilon_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}; \quad \gamma_{r\theta} = \left\{\frac{\partial u_\theta}{\partial r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r}\right\}$$

Reduced Stress-Strain Relation:

$$\sigma = \mathbf{D}\varepsilon \quad \text{where} \quad \mathbf{D} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}$$

Auxiliary Equation: The through-thickness strain is given by

$$\varepsilon_{zz} = -\frac{\nu}{E}\left(\sigma_{rr} + \sigma_{\theta\theta}\right)$$

from which one can obtain the out-of-plane displacement u_z via integration: $u_z = \int \varepsilon_{zz}\, dz$.

The Airy Stress Function $\Phi(r,\theta)$: Defining an Airy stress function through:

$$\sigma_{rr} = \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2}; \quad \sigma_{\theta\theta} = \frac{\partial^2 \Phi}{\partial r^2}; \quad \sigma_{r\theta} = \frac{1}{r^2}\frac{\partial \Phi}{\partial \theta} - \frac{1}{r}\frac{\partial^2 \Phi}{\partial r \partial \theta}$$

The governing differential equation for plane stress then becomes:

$$\nabla^4 \Phi \equiv \left\{\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right\}\left\{\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right\}\Phi = 0.$$

THE EQUATIONS OF LINEAR ELASTICITY IN CYLINDRICAL POLAR COORDINATES: PLANE STRAIN

Displacements: $\mathbf{u}^{\mathrm{T}} = \left[u_r(r,\theta) \quad u_\theta(r,\theta) \right]$

Strains: $\varepsilon^{\mathrm{T}} = \left[\varepsilon_{rr}(r,\theta) \quad \varepsilon_{\theta\theta}(r,\theta) \quad \gamma_{r\theta}(r,\theta) \right]$

Stresses: $\sigma^{\mathrm{T}} = \left[\sigma_{rr}(r,\theta) \quad \sigma_{\theta\theta}(r,\theta) \quad \sigma_{r\theta}(r,\theta) \right]$

Equilibrium Equations: (static equilibrium, no body forces):

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0$$

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{2\sigma_{r\theta}}{r} = 0$$

Strain-Displacement Relations:

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}; \quad \varepsilon_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}; \quad \gamma_{r\theta} = \left\{ \frac{\partial u_\theta}{\partial r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} \right\}$$

Reduced Stress-Strain Relation:

$$\sigma = \mathbf{D}\varepsilon \text{ where } \mathbf{D} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \nu/(1-\nu) & 0 \\ \nu/(1-\nu) & 1 & 0 \\ 0 & 0 & (1-2\nu)/2(1-\nu) \end{bmatrix}$$

Auxiliary Equation: The through-thickness normal stress needed to obtain a state of plane deformation is:

$$\sigma_{zz} = \nu(\sigma_{rr} + \sigma_{\theta\theta})$$

The Airy Stress Function $\Phi(r,\theta)$: Defining an Airy stress function through:

$$\sigma_{rr} = \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2}; \quad \sigma_{\theta\theta} = \frac{\partial^2 \Phi}{\partial r^2}; \quad \sigma_{r\theta} = \frac{1}{r^2}\frac{\partial \Phi}{\partial \theta} - \frac{1}{r}\frac{\partial^2 \Phi}{\partial r \partial \theta}$$

The governing differential equation for plane stress then becomes:

$$\nabla^4 \Phi \equiv \left\{ \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \right\} \left\{ \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \right\} \Phi = 0.$$