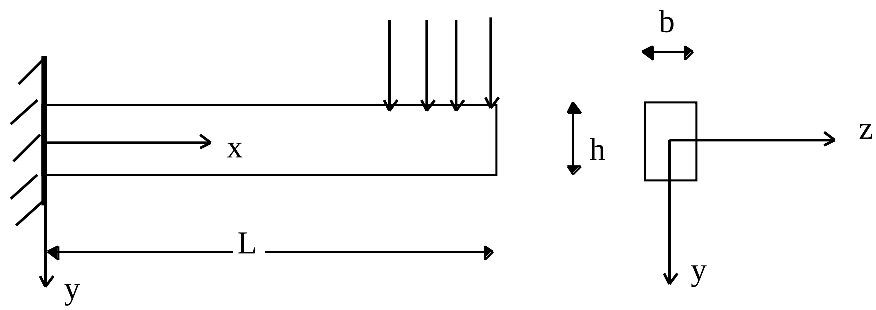
BENDING OF BEAMS

5.1 BENDING OF BEAMS

Beams are thin long structures that are *transversely* loaded. Consider the cantilever beam shown in the figure. What are the stresses, strains and displacements of this structure? In principle, we can try to solve the complete equations of elasticity subject to appropriate boundary conditions. This is however rather complicated. So, we will develop an approximate and simpler theory.



5.2 BERNOULLI-EULER BEAM BENDING THEORY:

Restrict attention to beams of vertically symmetric cross-section, where the transverse (y-directed) loading is in the plane of symmetry. Experimentally it is found that:

- (a) The only predominant quantities are the bending stress σ_{xx} and the bending strain ε_{xx} with all the other stress and strain components being negligible in comparison to these. Note that this cannot strictly be true because we certainly should have some normal stress near the top surface of the beam due to the applied loads! Again, by considering a cut through any cross-section, we would expect some shear stresses on these faces to balance the applied y-directed loads. Also, if we have a stress σ_{xx} , then we would expect some contraction or expansion in the y- and z-directions from the Poisson effect. What we are claiming here is that these are not as important as the bending stress and bending strain, and so we are going to throw everything but these out from our theory. This means that our theory is only approximately valid, and is actually not "clean". We will have to pay for these assumptions soon.
- (b) The centroidal surface along the length of the beam is longitudinally unstressed and unstretched, and therefore it is called the **neutral** surface. We can intuitively guess that for the cantilever beam shown above, any longitudinal (x-directed) line element will want to stretch at the top, and will compress at the bottom. Therefore the top is in tension and the bottom is in compression for the beam shown. And therefore there must

be a transition plane where there is no normal stress! This is true for all beams, not just the one shown in the figure. We locate our coordinate system such that the xz-plane is on this neutral centroidal surface.

· (c) All quantities are essentially independent of 'z'; ie, they do not vary in the thickness direction.

Mathematically, the above can be stated as:

$$\varepsilon_{yy} = \frac{\partial u_y}{\partial y} = 0 \tag{1}$$

$$\varepsilon_{xy} = \frac{1}{2}\left\{\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right\} = 0 \tag{2}$$

and we can neglect all the other strain components. Also, since only the bending stress σ_{xx} is predominant, we can neglect all the other stress components. From the stress-strain relations, we have:

$$\varepsilon_{xx} = \frac{1}{E}\,[\sigma_{xx} - \upsilon\,\underbrace{(\sigma_{yy} + \sigma_{zz})}_{\textbf{negligible}}] \approx \frac{\sigma_{xx}}{E} \tag{3}$$

where E is the Young's modulus and ν is the Poisson's ratio of the linear elastic isotropic material of the beam.

Now let us look at the consequences of our assumptions; they simplify the equations of elasticity considerably.

- From (1): We immediately recognize that u_y is not a function of 'y' and since nothing varies with 'z' according to our assumptions, we have:

$$\boxed{u_y = u_y(x)} \tag{4}$$

Physical meaning: every point on a cross-section (ie, at a given x-location) has the same vertical displacement u_y, which by the way is called the **beam deflection**. This beam deflection will be our favorite creature, and we will now try to cast everything else in terms of this.

- From (2):

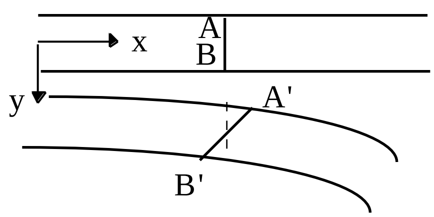
$$\frac{\partial u_x}{\partial y} = -\frac{\partial u_y}{\partial x} = -\frac{du_y(x)}{dx}$$ since u_y depends only on one variable 'x'.

$$u_x = -\int \frac{du_y(x)}{dx}\, dy = -y\frac{du_y(x)}{dx} + \underbrace{f_1(x)}_{\text{function of integration}}$$

However, since the neutral axis is unstretched, we set $u_x(y=0) = 0$, and so $f_1(x)=0$; and we are rid of it! Therefore:

$$\boxed{u_x = -y\frac{du_y(x)}{dx}}. \qquad (5)$$

Physical meaning: The longitudinal displacement u_x varies linearly with the y-distance from the neutral axis. Along with (4) this means that any cross-sectional plane stays a plane, though it displaces and rotates a bit from its original position.



- The bending strain using (5) becomes:

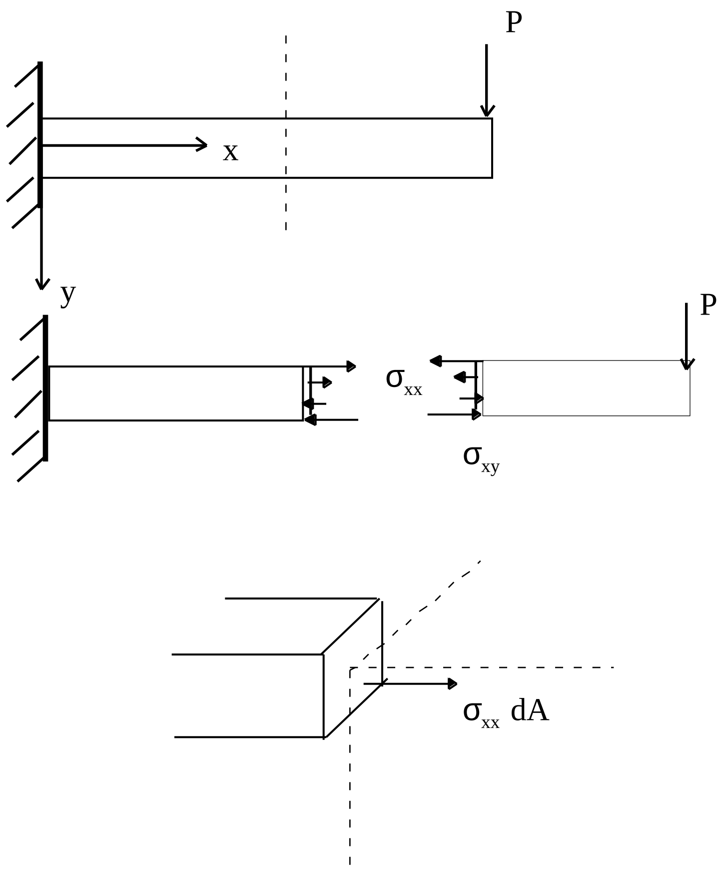
$$\boxed{\varepsilon_{xx} = \frac{\partial u_x}{\partial x} = -y\frac{d^2u_y(x)}{dx^2}} \qquad (6)$$

-The bending stress, from (3) and (6) is:

$$\boxed{\sigma_{xx} = E\varepsilon_{xx} = -E\ y\frac{d^2u_y(x)}{dx^2}} \qquad (7)$$

And with that we have cast all the quantities of interest (the bending stress σ_{xx}, the bending strain ε_{xx}, and the longitudinal displacement u_x) in terms of the beam deflection u_y.

We have still not used Newton's laws. Now is where we have to pay up for all of our not-so-clean approximations. Turns out that the equations of equilibrium that we have derived previously cannot be satisfied in general since we have neglected all but the bending stress. Therefore all we can do here is to impose equilibrium globally and in an average sense.



Cut the cantilever beam shown in figure at some cross-sectional location and look at the free-body diagram of either side. Clearly, we need the bending stress σ_{xx} to balance the moment caused by the applied loads, and we need a shear stress σ_{xy} for force balance in the y-direction. Leaving aside the force balance in the y-direction, since our approximate theory cannot deal with the shear stress, let us look at force balance in the x-direction, and also moment balance.

- From force balance in the x-direction, we must have:

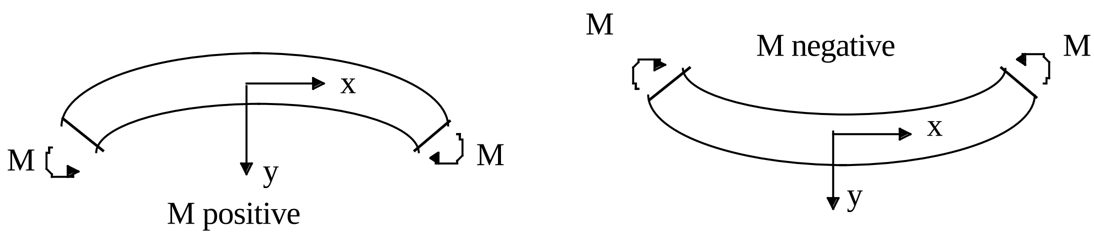
$$\int \sigma_{xx}\; dA = \int \left\{ -Ey \frac{\partial^2 u_y(x)}{\partial x^2}\; dA \right\} = -E \frac{\partial^2 u_y(x)}{\partial x^2} \underbrace{\int y\; dA}_{=0} = 0.,$$ (there is no net axial force)

In the above, we recognize that the first moment of area about the centroidal axis is zero (that is the *definition* of the centroid).

- Next consider moment balance. The net moment caused by the bending stress should balance the net moment due to any applied y-directed forces. Thus:

$$\int y \ \sigma_{xx} \ dA = \int \left\{ -Ey^2 \frac{\partial^2 u_y(x)}{\partial x^2} \ dA \right\} = -E \frac{\partial^2 u_y(x)}{\partial x^2} \underbrace{\int y^2 \ dA}_{=I_z} = -M(x)$$

where $I_x = \int y^2 \ dA$ is the second moment of area (also called the area moment of inertia) of the cross-section about the centroidal z-axis. M(x) is the net cross-sectional moment due to the applied loads, and I have introduced a sign convention for moments as follows:



A moment is treated as positive if it causes compression of the beam at positive y-locations. {A handy way to remember this: With xy-coordinate as shown, a smiling beam is a negative thing, and a frowning beam is a positive thing. Everything is upside down in the Bernoulli-Euler universe! Remember our y-axis points downwards too!}

We now have a tool to figure out the deflection of a beam. We simply integrate:

$$\boxed{EI_z \frac{d^2 u_y}{dx^2} = M(x)} \qquad \text{...Bernoulli-Euler beam equation} \qquad (8)$$

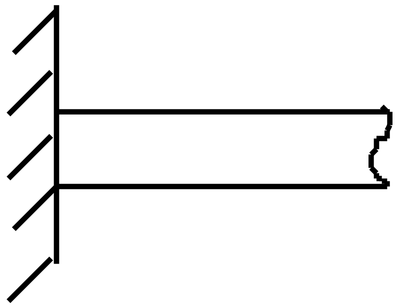
To do so, we first need to compute the net cross-sectional moment for any given beam, and we also need boundary conditions for the beam deflection. Before we do that, note that the bending stress can now be written directly in terms of the bending moment using the above:

$$\boxed{\sigma_{xx} = -Ey \frac{d^2 u_y}{dx^2} = -\frac{M(x)y}{I_z}}$$

5.3 BOUNDARY CONDITIONS:

Beams can be supported in several ways. Two common supports are:

(i) Fixed-end (clamped end, built-in end): This is when a beam is embedded in a rigid wall.



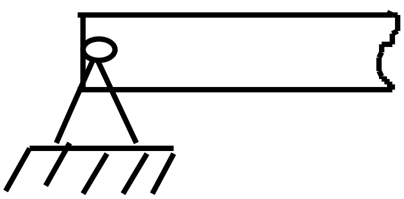
The displacements at a fixed end are zero. This means that:

$$u_y\Big|_{\text{fixed-end}} = 0$$

$$u_x\Big|_{\text{fixed-end}} = -y\frac{du_y}{dx}\bigg|_{\text{fixed-end}} = 0 \qquad \frac{du_y}{dx}\bigg|_{\text{fixed-end}} = 0$$

In words, the beam deflection and the slope are zero at a fixed end. Note that a fixed end can resist applied forces and moments.

(ii) Simply-supported end (hinged end, pinned end): This is when the beam is supported by a hinge whose pin goes through the neutral axis. The vertical displacement at a hinged end is zero. Also, note that a hinge cannot resist a moment. This means that:



$$u_y\Big|_{\text{simple-support}} = 0$$

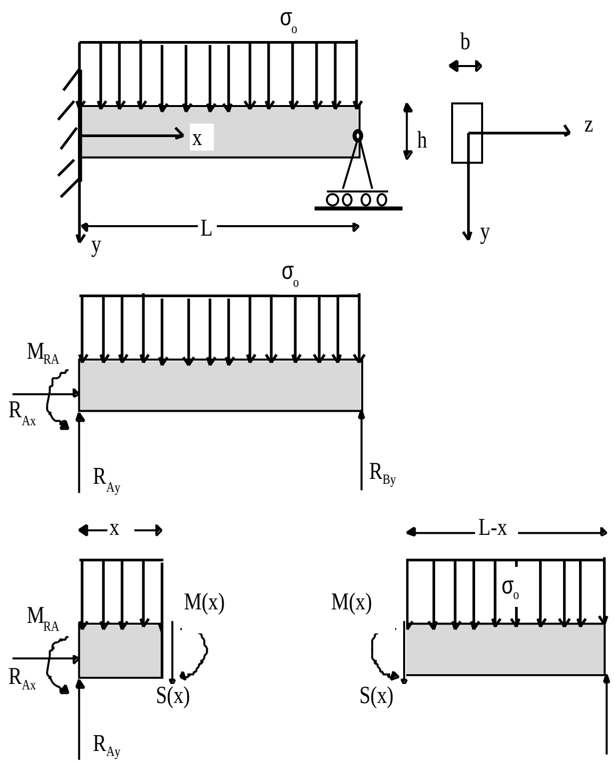
$$M\Big|_{\text{simple-support}} = EI_z\frac{d^2u_y}{dx^2}\bigg|_{\text{simple-support}} = 0 \qquad \frac{d^2u_y}{dx^2}\bigg|_{\text{simple-support}} = 0$$

Sometimes, a simple-support will be shown on rollers so that horizontal displacements even at the neutral axis are possible (Note that this does not mean that the neutral axis stretches). This does not change the boundary condition for the beam deflections at all.

5.4 SOLUTION PROCEDURE:

We can now make the analysis of any beam a routine thing by following a set of steps best explained by means of an example.

Example: Consider a fixed-hinged beam of uniform **flexural rigidity** (the technical name for the combination EI_z) which carries a uniform distributed load on the top of σ_o N/m^2.



Step 1: Look at the free-body diagram of the entire beam by removing it from its supports. Note that the problem is statically indeterminate since the unknown reactions are $\{M_{RA}, R_{Ax}, R_{Ay}, R_{By}\}$ and we can have at most two force balance and one moment balance equation.

Step 2: Make a cut at some cross-section at a distance x from the left end. From the free-body diagram of the right part of the beam, we have:

$$M(x) = -R_{By}[L-x] + \frac{\sigma_o b}{2}[L-x]^2$$

.

Using this in the Bernoulli-Euler equation:

$$EI_z\frac{d^2u_y}{dx^2} = -R_{By}[L-x] + \frac{\sigma_o b}{2}[L-x]^2$$

which upon integrating twice yields:

$$EI_z u_y = -R_{By}\left[\frac{Lx^2}{2} - \frac{x^3}{6}\right] + \frac{\sigma_o b}{2}\left[\frac{L^2x^2}{2} + \frac{x^4}{12} - \frac{Lx^3}{3}\right] + c_1x + c_2$$

Using the boundary conditions: $u_y\big|_{x=0} = 0$; $\left.\frac{du_y}{dx}\right|_{x=0} = 0$; $u_y\big|_{x=L} = 0$ we find that $c_1 = c_2 = 0$

and $R_{By} = \frac{3}{8}\sigma_o bL$, and so:

the beam deflection is: $u_y(x) = \frac{\sigma_o bL^4}{48EI_z}\left\{\frac{x}{L}\right\}^2\left\{2\left(\frac{x}{L}\right)^2 - 5\left(\frac{x}{L}\right) + 3\right\}$

We can of course get the bending stress and the other quantities of interest once we have the beam deflection. We can also figure out where the beam deflection is largest and so on. You will do such things in a homework assignment.